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On the Normalization of Dimensioned Variables

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Abstract

We attempt to resolve the central dispute in recent debates on dimensional consistency of economic and ecological variables in ecological economics (Mayumi and Giampietro, 2010; Malghan, 2011; Chilarescu and Viasu, 2012; Baiocchi, 2012). Using canonical examples from ecology and economics, we demonstrate that the well-established procedures of normalization and nondimensionalization can be used to circumvent the technical problem of dimensional consistency of ecological economics models. However, we also show how normalization does not directly address the problem of mapping between cardinal and ordinal variables that is the primary source of dimensional consistency problems in ecological economics.

Key words: Dimensional consistency, Normalization and nondimensionalization, Cardinal and ordinal variables

1. Introduction

The fundamental concern of ecological economics is to accurately model all aspects of the economy-ecosystem interaction problem — the myriad ways in which the economic and ecological systems are connected to each other. Almost all the monetary and physical variables used to describe economy-ecosystem interaction are dimensional in nature. The exact cardinal value taken by dimensioned variables is contingent on the particular measurement unit used. While several recent papers on the subject have pointed to the care required in using dimensioned variables in ecological economics, there is little consensus on how dimensional variables must be incorporated in economy-ecosystem interaction models (Mayumi and Giampietro, 2010; Malghan, 2011; Chilarescu and Viasu, 2012; Baiocchi, 2012; Mayumi and Giampietro, 2012). Mayumi and Giampietro (2010) inaugurated the debate by making the provocative claim that many models in economics and ecological economics that make use of transcendental functions like the logarithm are fundamentally flawed when these functions use what are apparently dimensioned variables. Malghan (2011) claimed that several popular biophysical sustainability indicators are dimensionally inconsistent because they neglect the 'qualitative residual' that is the defining characteristic of any social-ecological system (Georgescu-Roegen, 1971). In a brief comment, Chilarescu and Viasu (2012) showed that the critique of a neoclassical production function (Arrow et al., 1961) on the basis of it being dimensionally inconsistent does not take into account the fact that parameters of a production function are dimensioned variables, too. Thus in the familiar Cobb-Douglas production function of the form $Y = F(K, L) = AK^{\beta}L^{\alpha}$, the parameter A has appropriate dimensions (contingent on α and β) such that the function itself has the exact same dimension as Y (Chilarescu and Viasu, 2012). In an earlier debate on a similar subject, Folsom and Gonzalez (2005) had showed how parameters of the Cobb-Douglas production function are assumed to have implicit dimensions required by dimensional consistency — in response to the dimensional inconsistency claim made by Barnett-II (2003). Baiocchi (2012) used examples from a variety of disciplines including the IPAT identity and Environmental Kuznets Curve to critique the Mayumi and Giampietro (2010) claim, and also offered a critical historical literature review of dimensional analysis.

Unfortunately, the recent debate on dimensional consistency in ecological economics has only helped to muddy the waters rather than provide a consistent framework for achieving dimensional consistency while studying the economy-ecosystem interaction problem. It is trivial to demonstrate that a logarithmic function cannot have dimensioned variables as its argument. The more pertinent question is whether it might be possible to nondimensionalize basic models of economy-ecosystem interaction that are of interest to ecological economists. We illustrate the problem with the transcendental logarithm function that has been at the center of the recent debate. In their rejoinder to Chilarescu and Viasu (2012), Mayumi and Giampietro use the familiar Maclaurian expansion of $\ln(1+x)$ and $\ln(1-x)$ to obtain a polynomial expansion for the natural logarithm of any positive real number, $z \in \mathbb{R}_+$ (Mayumi and Giampietro, 2012, Eq-5):

$$\ln(z) = 2\left\{\left(\frac{z-1}{z+1}\right) + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \dots\right\} \tag{1}$$

It is straightforward to see that equation-1 cannot take a dimensioned z. Thus, Mayumi and Giampietro (2012) argue that a regression model that includes a term like $\ln\left(\frac{V}{L}\right)$ used by Arrow et al. (1961) in their labor-capital substitution model is problematic because the logarithm takes on a dimensioned quantity (measured in US dollars per person-year of labor unit, for example). Even in the 1960s, the classic paper by Arrow et al. (1961) had been critiqued for not considering the dimensional consistency of production function specifications (De Jong, 1967; De Jong and Kumar, 1972; Cantore and Levine, 2012). However, Mayumi and Giampietro (2010, 2012) ignore the fact that it is possible in theory to obtain non-dimensional versions of V and L though the well-established process of normalization. In principle, there should be no objection to using an expression like $\ln\left(\frac{V}{L}\right)$ if value added (V) and quantity of labour (L) are expressed as non-dimensional variables.

While Mayumi and Giampietro (2012) cite several examples from prominent economists committing the apparent error of using dimensioned quantities in the logarithmic functions, we demonstrate in Section-2 that normalization or nondimensionalizatoin can in principle address this problem. We argue that this is a relatively minor technical point, and that the more fundamental problem is that of representing the economy-ecosystem interaction problem in a dimensionally consistent fashion. The remainder of this paper is organised as follows: the next section will review normalization and nondimensionalization using canonical examples from economics and ecology. It is not merely sufficient for an ecological economics model to be dimensionally consistent. The key question is *'whether or not the selected dimensional choice for a given expression has an operational meaning or relevance for the purpose [of] analysis'* (Mayumi and Giampietro, 2012, emphasis in original). While this was the true import of Mayumi and Giampietro (2010), the subsequent papers in the debate have missed the forest for the trees by focussing exclusively on narrow technical dimensional consistency. In Section-3 we discuss the limitations of normalization and nondimensionalization procedures. In particular, we show that it is nontrvial to normalize dimensioned variables in analytically accurate models of economy-ecosystem interaction.

2. Normalization and Nondimensionalization

Using several canonical (and elementary) examples from ecology and examples, we demonstrate in this section that normalization and nondimensionalization can address dimensional consistency issues in ecological economics. We examine the production function and the consumer's utility maximization problem from elementary microeconomics; the logistic population growth model from ecology; and the normalisation of the Gaussian distribution in statistics.

2.1. Normalization of the Canonical Cobb-Douglas Production Function

The standard Cobb-Douglas production function for two inputs K and L can be represented as:

$$Y = AK^{\beta}L^{\alpha} \tag{2}$$

Now consider a simple constant-returns version of equation-2 such that ($\beta = 1 - \alpha$) and ($0 < \alpha < 1;\ K, L > 0$):

$$Y = AK^{1-\alpha}L^{\alpha} \tag{3}$$

The central dimensional concern with the Cobb-Douglas function in equation-3 is that capital (K), and labor (L) are measured in units that are different from each other, and from the output (Y). The constant A has a dimension that is contingent on the factor-share parameter, α such that equation-3 is dimensionally consistent. To make this point of A being dimensional even more explicit, equation-3 can be rewritten as:

$$Y = (A_K K^{1-\alpha})(A_L L^{\alpha}) \tag{4a}$$

$$A = A_K A_L \tag{4b}$$

where the dimensional constants A_K and A_L are the so-called efficiency parameters. The presence of these two dimensioned quantities makes analytical work and interpretation difficult. However, as shown by De Jong (1967) and Cantore and Levine (2012), equation-4 is most easily normalised and rendered into a non-dimensional form. Consider a normalization-point Y_0 such that:

$$Y_0 = (A_K K_0^{1-\alpha})(A_L L_0^{\alpha}) \tag{5}$$

Now dividing equation-4 by equation-5 we readily obtain the non-dimensional version of the constant-returns Cobb-Douglas function:

$$y = k^{1-\alpha}l^{\alpha} \tag{6a}$$

$$y = \frac{Y}{Y_0};\ k = \frac{K}{K_0};\ l = \frac{L}{L_0} \tag{6b}$$

Any econometric models involving logarithms of the non-dimensional variables (y, k, l) will pose no dimensional issues — for example a log-log model to estimate factor share, α. While equation- 6 eliminates the dimensional constants, it offers no clarity on how to pick the normalization point (Y_0). In the context of a neoclassical economic growth model, it would be most intuitive to use the steady state value as the normalization point. While the choice of normalization point is easily determined for the present problem, we will show below how this can be non-trivial when studying the economy-ecosystem interaction problem. Before we take up another canonical example — the logistic growth equation from ecology to illustrate the process of nondimensionalization (a homologue of the normalization process discussed here) — it is important to note that more general production functions (CES, for example) can be normalized in the same manner as the pedagogically simple case of Cobb-Douglas discussed here (Klump and La Grandville, 2000; Klump and Saam, 2008; Cantore and Levine, 2012; Temple, 2012).

2.2. The Logistic Equation

Consider the Logistic Equation that has been the pedagogical model of choice for students of ecology from the time Alfred Lotka formalised the original Verhulst formulation in the context of population growth of parasite colonies (Lotka, 1925). The simple population growth logistic equation with a fixed carrying capacity, K and population growth rate, r can be written as:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \frac{P}{K}\right) \; ; \quad P(0) = p_0 \tag{7}$$

where P is the population at any time t; and the initial population is known such that $P(0) = P_0$. In the above equation, all the four variables are dimensional — P and K have the dimension of [**N**] (number of individual plasmodium parasites in a colony for example); t has the [**T**] dimension (time, measured in hours or minutes); and r has the dimension of $[\mathbf{T^{-1}}]$ (inverse time dimension, measured in per-hour or per-minute, persevering with the plasmodium colony growth example). The units in which population and time are measured are arbitrary and the parameter values in equation-7 will change if we went from measuring time in hours to say, in minutes or years.

It is straightforward to nondimensionalize equation-7 so that it is invariant to particular choices of units for population and time. This is achieved by scaling or normalizing the time and population variables as:

$$\tau = \frac{t}{\left(\frac{1}{r}\right)} \tag{8a}$$

$$x = \frac{P}{K} \tag{8b}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\mathrm{d}\,(Kx)}{\mathrm{d}\,(\tau/r)} = rK\frac{\mathrm{d}x}{\mathrm{d}\tau} \tag{8c}$$

$$x_0 = \frac{P_0}{K} \tag{8d}$$

The new variables x and τ defined in equation-8 are non-dimensional. Substituting equation-8 in equation-7 we obtain the nondimensionalized form of of the logistic equation:

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = x\,(1-x)\ ;\ \ x(0) = x_0 \tag{9}$$

Unlike the original dimensioned variables, P, K, t, and r, the scaled non-dimensional variables x and τ can be used in any transcendental functions like the natural logarithm or the exponential function. Like any nondimensionalization process, the scaled variables x and τ are related to the *intrinsic* property of the physical phenomenon being studied. The scaled population, x represents the population relative to the carrying capacity, K and is the intrinsic unit for measuring population in a simple logistic model.[1] By measuring population using non-dimensional x, we have scaled the problem so that equation-9 applies to a wide variety of phenomena following the logistic growth pattern. Similarly $1/r$ that we used to scale time, t to obtain the non-dimensional τ is the intrinsic unit for measuring time in the context of population growth models. In an exponential growth model (the initial part of the logistic growth curve when $P \ll K$), the population grows by a factor of e in the time

[1] One could have also carried out the nondimensionalization of equation-7 by setting $x = P/P_0$. A non-dimensional x that is a scaled by the initial population is however not intrinsic to the system as the carrying capacity (for a system with time invariant K).

interval $1/r$ – an *intrinsic* unit for measuring time in any exponential growth problems. Besides being an intrinsic representation, the scaled non-dimensional form of the logistic equation is also the most parsimonious representation of the problem of carrying capacity constrained growth.

2.3. The Standard Normal

Consider a random variable X that is distributed with mean μ and variance σ^2 ($X \sim \mathcal{N}(\mu, \sigma^2)$). The Gaussian distribution for X, $f(X)$ is given by:

$$f(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \tag{10}$$

Any normally distributed variable can be expresses in terms of the standard normal, Z (where $Z \sim \mathcal{N}(0, 1)$).

$$f(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} \tag{11}$$

As every beginning student of statistics is taught, for any random variable $X \sim \mathcal{N}(\mu, \sigma^2)$, $Z = \frac{X-\mu}{\sigma}$ is a standard normal, or $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$. Besides helping with statistical inference, this normalization process is of significance for our discussion about dimensioned variables. X is a dimensioned variable (has the dimensions of $[\mathbf{T}]$ for example if X was measuring some temporal phenomenon). However, the normalized variable Z is dimensionless as μ and σ have the same dimensions as X –$[\mathbf{T}]$ in the present example. Thus, while X cannot be used as an argument in transcendental functions, an expression of the form $Y = \ln(Z)$ can be evaluated using equation-1. This normalization process is even more significant if one considers the fact that the sum of a sufficiently large set of independent random variables (with finite variance) will converge to a normal distribution (the Central Limit Theorem).

2.4. The Numéraire Good and Consumer's Utility Maximization Problem

The most widely used example of normalization in economics – by a wide margin – is the numéraire good. All prices in the pure theory of exchange are relative prices — prices that have been normalised by an appropriate numéraire. Money (dollars for example) is simply the most common choice for the numéraire. In principle, any other commodity can be used as a numéraire.

Consider an individual's utility function defined by a Cobb-Douglas function (in a simple two-good case) as follows:

$$U = X^{1-\alpha} Y^{\alpha} \tag{12}$$

Following our discussion in equation-5, we can write out a corresponding utility normalization point as:

$$U_0 = X_0^{1-\alpha} Y_0^{\alpha};\ X_0, Y_0 > 0 \tag{13}$$

Dividing equation-12 by equation-13 we obtain a non-dimensional analogue of equation-6:

$$u = x^{1-\alpha} y^{\alpha} \tag{14a}$$

$$u = \frac{U}{U_0};\ x = \frac{X}{X_0};\ y = \frac{Y}{Y_0} \tag{14b}$$

All three variables (utility, and the quantity of two goods that are consumed) in equation-14 are non-dimensional. While $\ln(U)$ is not defined, equation-1 can be used to evaluate $\ln(u)$. Before we consider the consumer's utility maximization problem, we write out the budget constraint faced by the consumer:

$$P_X X + P_Y Y \leq M \tag{15}$$

In equation-15, M is the disposable income available to the consumer; and P_X and P_Y are respectively prices (say in dollars per unit) of goods X and Y respectively. The budget constraint when expressed using dimensionless x and y (instead of dimensioned quantities X and Y) can be written out as:

$$\tilde{P_X} x + \tilde{P_Y} y \leq M \tag{16a}$$

$$\tilde{P_X} = X_0 P_X \tag{16b}$$

$$\tilde{P_Y} = Y_0 P_Y \tag{16c}$$

In equation-16 $\tilde{P_X}$ and $\tilde{Y_0}$ are simply prices corresponding to normalised (and dimensionless) quantities of X and Y. Money, measured in dollars ($) is the numéraire in both equations (15) and (16). While P_X and P_Y have the dimensions of $\frac{dollars}{quantity}$, $\tilde{P_X}$ and $\tilde{P_Y}$ have dimensions of $dollars$. One of the fundamental insights from consumer's problem is that the neither the budget set nor the budget constraint is affected by our choice of numéraire. Now, if we normalize equation-16 using x as the numeraire good, we can rewrite the budget constraint as:

$$x + \bar{P_Y} y \leq \bar{M} \tag{17a}$$

$$\bar{P_Y} = \frac{\tilde{P_Y}}{\tilde{P_X}} \tag{17b}$$

$$\bar{M} = \frac{M}{\tilde{P_X}} \tag{17c}$$

Every term in the budget constraint represented by equation-17 is dimensionless. As the relative price $\bar{P_y}$ and $\bar{M}$ are dimensionless they can be used as arguments in a transcendental function. Thus a regression equation that uses $\ln(\bar{M_i})$ poses no dimensional problems (where $\bar{M_i}$ the disposable income of household i).

We can now write out the consumer's utility maximization problem using equations (14) and (17). The dimensionless Lagrangian is simply:

$$\mathcal{L} = \left(x^{1-\alpha} y^{\alpha}\right) + \lambda \left(x + \bar{P_Y} y - \bar{M}\right) \tag{18}$$

By setting $\frac{\partial \mathcal{L}}{\partial x} = 0$, $\frac{\partial \mathcal{L}}{\partial y} = 0$; and eliminating λ we obtain the dimensionless first order condition for the consumer's utility maximization problem:

$$\left(\frac{1-\alpha}{\alpha}\right) \frac{y}{x} = -\left(\frac{1}{\bar{P_Y}}\right) \tag{19}$$

Every single variable in equation-19 is dimensionless.

3. Object Lessons

We have demonstrated using canonical examples from economics, ecology, and statistics that normalization and nondimemsionalization can transform dimensional forms into their dimensionless counterparts. The examples presented in the previous section show that in theory, normalization can circumvent the objections raised by Mayumi and Giampietro (2010) in the recent debate over dimensions. However, as Mayumi and Giampietro (2012) point out in their rejoinder, the more relevant question is one of delineating the physical basis for normalization. In the examples that we have considered, the nondimensionalization procedure for the logistic equation or the construction of the standard normal statistic is well-grounded. From the two economics examples we have considered, normalization using an arbitrary choice of the numeraire good in the consumer problem is well-established. A production function on the other hand must not only be dimensionally consistent but also reflect the physical basis for production. Normalization only solves the technical problem of dimensional consistency but the normalised representation of the production process is only as good as the original dimensioned representation. An accurate physical representation of the production process has been one of the founding tenets of ecological economics (Georgescu-Roegen, 1971; Kraev, 2002; Røpke, 2004).

As an illustration of the difficulties involved in selecting a normalization point in realistic models of economy-ecosystem interaction, consider any model that includes a throughput variable ($\dot{X}$), say measured in kilograms per year so that $\dot{X}$ has the dimensions of $[\mathbf{MT^{-1}}]$. The throughput $\dot{X}$ cannot be an argument in any transcendental function. It is trivial to normalize the throughput with some reference throughput, $\dot{\mathbf{X}}$ to obtain a nondimensional version $\dot{x} = \frac{\dot{X}}{\dot{\mathbf{X}}}$ such that the normalized throughput, $\dot{x}$ has no physical dimensions and can be used as arguments in transcendental functions. Indeed, such a measure is homologous to the rapidity measure used in in physics to characterize speed relative to the speed of light.[2] Unlike relativity-physics however, the choice of reference throughput, $\dot{\mathbf{X}}$ is not universal but

[2] In physics, rapidity, ϕ is defined as $\phi = \tanh^{-1}\left(\frac{v}{c}\right)$ where c is the speed of light.

highly context dependent. A possible candidate for reference throughput is the maximum sustainable throughput — the throughput above which the integrity of the underlying biophysical system in jeopardy. Consider an illustrative example — throughput of timber from a forest. The maximum sustainable throughput is a function of the health of the underlying forest ecosystem and will vary across both space and time. A tropical forest will necessarily have a different maximum sustainable throughput from a temperate forest. Even in a single location, maximum sustainable throughput will vary with time. The determination of maximum sustainable throughput is a function of ecosystems as funds rather than stocks (Georgescu-Roegen, 1971; Malghan, 2011).

The difficulty with determining an appropriate normalization point in the throughput example above is related to a more general problem of mapping ordinal and cardinal variables in a dimensionally consistent fashion. An accurate representation of the economy-ecosystem interaction problem requires accounting for ecosystem as a fund in addition to ecosystem as simply a collection of stocks. Unlike stocks and flows, funds and fluxes are ordinal and subject to additional dimensional consistency constraints. The mapping between an ordinal fund-flux space and the cardinal stock-flow space is at the heart of economics of ecosystem services (Farley, 2012; Malghan, 2011). In the current debate of dimensioned variables in ecological economics, the true import of the first salvo fired by Mayumi and Giampietro (2010) was the fact that several empirical models are not careful about making the distinction between fund and stock functions of ecosystem. We would be missing the forest for the trees if we focused the dimensions debate exclusively on technical aspects of normalization. Normalization procedure, as demonstrated using elementary examples is well-established for cardinal variables but the cardinal stock-flow space alone is inadequate for accurately modelling the economy-ecosystem interaction problem. There is a need for ecological economics to develop models of economy-ecosystem interaction that are at once realistic representation of the problem and are dimensionally consistent.

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