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Optimal Pricing and Advertising Policies for Bottom of Pyramid Markets: An Analytical Approach

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Abstract

We propose a model for developing optimal decisions and normative policies for pricing and advertising of products/services to markets at the 'bottom of pyramid (BOP).' This concept has been popularized in the recent times by Prahalad (2006). Our model considers two types of market segments. The first type is the bottom of pyramid market, which is large in size, but has limited ability to pay. The second type of the market is smaller in size, but can pay higher prices. The product/service offered to the two markets is differentiated in such a way that the base product (of appropriate quality) is available to the BOP market, while the premium product at a higher price is available to the higher-end market. The two markets are linked to each other such that there is a positive effect of customer base in the BOP market on the diffusion of product in the premium market. Successful practices of such kind of models have been reported in widely documented healthcare case studies such as Aravind Eye Care (Rangan 1993). The product diffusion in the two markets is modeled using a pure innovation model by Fourt and Woodlock (1960). Using optimal control methodology, we derive pricing and advertising policies for two types of organizations - for-profit organization (FPO) and non-profit organization (NPO). Thus, our analytical research design follows a 2 x *2* x *2* x *2 (markets – BOP vs. Premium, strategies - pricing and advertising, organizations – FPO vs. NPO, and modeling – static vs. dynamic) design.*

Our optimal normative policy results can be summarized as follows: (i) A non-profit organization (NPO) charges lesser price per unit in both BOP and Premium markets, as compared to the for-profit organization (FPO), (ii) A non-profit organization (NPO) spends equal amount of money in advertising or promoting the product/service as that spent by a for-profit organization (FPO), (iii) A FPO charges lesser price per unit in the BOP markets as compared to the Premium market, (iv) The FPO receives lesser contribution margin per unit in the BOP market, as compared to the Premium market, (v) For the FPO, the ratio of advertising/promotion done in BOP market to that in the Premium market is governed by the parameters such as relative advertising effectiveness, cost of advertising, and contribution margin per unit in the two markets, (vi) Our dynamic pricing policy results for a FPO show that the prices are gradually increasing in the Premium market, and gradually decreasing in the BOP market, albeit after a threshold level of sales. The dynamic advertising policy results for a FPO show that advertising should gradually be decreased in the BOP market, but should remain stable in the Premium market. The NPO dynamic pricing and advertising results are similar to their static counterparts, though at much lower price levels. .

[Key words: Product diffusion models, Optimal control, Marketing policies]

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1 Introduction

The concept of “bottom of pyramid (BOP)” market by Prahalad (2006) has gained increased importance since its introduction. In defining the BOP market, Prahalad (2009, p. 2) clarifies:

> There is a lot of discussion on what the Bottom of the Pyramid market is and who constitutes that market. The original definition of the Bottom of the Pyramid was based on a simple premise. The concept was originally introduced to draw attention to the 4-5 billion poor who are unserved or underserved by the large organized private sector, including multinational firms. This group, until recently ignored by the private sector, could be a source of much needed vitality and growth. The assumptions influencing the focus on the Bottom of the Pyramid were also explicit: “Four billion poor can be the engine of the next round of global trade and prosperity. Serving the Bottom of the Pyramid consumers will demand innovations in technology, products and services, and business models. More important, it will require large firms to work collaboratively with civil society organizations and local governments. Market development at the Bottom of the Pyramid will also create millions of new entrepreneurs at the grass root level—from women working as distributors and entrepreneurs to village level micro enterprises.

We consider a situation which typifies marketing decision in this scenario and has gained increasing relevance in recent times. In such situations, a product or service is made available in two variants to two different markets. The first market is the BOP market, which is characterized by its large size, but with limited ability to pay. The second type of the market, which we label as the Premium market, is smaller in size, but can pay higher prices. The product/service offered to the two markets is differentiated in such a way that a base product (of appropriate quality) is available to the BOP market, while a higher-end product at a higher price is available to the Premium market. This segmentation view is supported in the literature in the context of base of pyramid by Rangan et al. (2011). The two markets are linked to each other such that there is a positive effect of customer base in the BOP market on the diffusion of product in the Premium market. The formation of customer base in the BOP market is akin to the corporate social responsibility (CSR) initiatives of organizations.

In this paper, we are interested in prescription or recommendation of the optimal approaches and policies for pricing and advertising to be adopted by marketers in such situations. We derive the pricing and advertising policies for two types of organizations - for-profit organization (FPO) and non-profit organization (NPO). Thus, our analytical research design follows a 2 x 2 x 2 x 2 (*markets* – BOP vs. Premium, *strategies* - pricing and advertising, *organizations* – FPO vs. NPO, and *modeling* – static vs. dynamic) design. To the best of our knowledge, no previous studies have addressed such modeling aspects related to BOP market.

Successful practices of such kind of management models have been reported in widely documented healthcare case studies such as Aravind Eye Care (Rangan 1993). With millions of patients served in the field of ophthalmic care, the Aravind Eye Care System in India has come to be called as McSurgery or in a way the McDonald's of cataract surgery in India: efficient, effective, scalable and sustainable (Stephen Miller, Wall Street Journal Online, August 5, 2006). Started by Govindappa Venkataswamy (popularly known as Dr. V) as an 11-bed clinic in 1976, it is today one of the largest eye-care systems in the world, catering largely to the poor in India.

A sense of compassion, commitment, and leadership are key elements of the Aravind model. The central principle that, productivity is fundamentally related to demand, makes it a viable business proposition. Volume brings down the cost and ensures the viability of the enterprise. Volume in turn is ensured by the combination of low cost, high quality and efficient procedures, as well as the appropriate use of technology.

Aravind Eye Hospitals try to maintain a specific ratio between paying and free patients, which keeps the enterprise financially viable. As a differentiator, the paying patients are offered a better set of post-surgical services. Integrating backward, a separate company has been set up to manufacture the intraocular lens, which helps provide quick and low cost cataract surgery. The model can be replicated, and some of its principles are universally applicable. These include: standardization, in-house manufacturing of affordable lenses, economies of scale, delegation of non-specialist tasks to less skilled workers, design of operations theater with two beds (one for the patient having surgery, the other for a patient being prepped), continuous process improvement, anticipation of demand, contingency planning in case of excess demand, and flexibility in terms of skill set of employees.

We first develop a static model for the modeling of BOP markets. This model has been developed based on the model of Weinberg (1983). However, Weinberg's (1983) model deals with only one market, and does not address the case of BOP markets. We then extend the static model to a dynamic model using the a basic model for new product /service innovation diffusion modeling approach proposed by Fourt and Woodlock (1960); and Bass (1969). To derive the optimal normative pricing and advertising policies, we use the optimal control methodology (Kamien and Schwartz, 1981; Sethi and Thompson, 2000). Several previous researchers have applied optimal control techniques to derive normative optimal policies in marketing context (e.g., Dockner and Jorgensen, 1988; Swami and Khairnar 2006; Swami and Dutta 2010). The current paper extends this stream of research in the situations of BOP markets as discussed above.

The rest of this paper is organized as follows. In Section 2, we present a review of the background literature in this area. In Section 3, we present the conceptual framework to motivate the discussion of the proposed mathematical models to represent the situation considered in this paper. In the next section, we present the static and dynamic models and their analyses. We conclude in Section 5 with discussion of future research areas.

2 Background Literature

In the following, we briefly review the areas of specific interest to the current research.

2.1 BOP Markets

In their seminal article, Prahalad and Hammond (2002) introduced the concept of BOP markets to draw attention to the 4-5 billion poor people of the world who are unserved or underserved by the large organized private sector, including multinational firms (Prahalad 2009). In his global bestseller book, *The Fortune at the Bottom of the Pyramid*, Prahalad (2006) asserts that the world's most exciting, fastest-growing new market is where one least expects it: at the bottom of the pyramid. By referring to several successful case studies from emerging markets such as India, Peru, Mexico, Brazil, and Venezuela, and industries such as salt, soap, banking,cell-phones, healthcare and housing, Prahalad (2006) strongly advocates that large MNC's can no longer afford to ignore the huge opportunity at the BOP markets. He shows that companies, which learn to serve these markets are not only making money, but are also helping millions of the world's poorest people escape poverty.

Since its introduction, considerable interest has been generated in the BOP proposition. Vachani and Smith (2008) discuss the issues related with distribution strategies for reaching the bottom of the pyramid. They highlight four major distribution problems in this context: (i) Poor road, communications, and electricity Infrastructure, (ii) Information problems, (iii) Lack of knowledge and skills, and (iv) Illiteracy.

Several authors undertook studies to understanding consumer behavior at bottom of pyramid. In support of BOP proposition, Subrahmanyan and Gomez-Arias (2008) find that despite income and resource constraints, BOP consumers are sophisticated and creative. They are motivated not just by survival and physiological needs but seek to fulfill higher order needs. Pitta et al. (2008) discuss the key elements involved in the BOP initiative: companies' motivations, characterization of the BOP consumers, and the business model to attend the BOP. Wood et al. (2008) recommend that MNC's wishing to successfully pursue BOP markets need to blend their understanding of BOP uniqueness, with a clear understanding of the other three concepts, namely share of heart, gobal umbrella brands and responsible marketing. Guesalaga and Marshall (2008) examine the purchasing power at the BOP. They analyzed secondary data on income, population, and expenditure at the BOP from different countries, and apply the buying power index (BPI) methodology to assess the purchasing power of low-income consumers. Their results also find support for the BOP proposition. De Silva et al. (2009) quantitatively measure the various influences on mobile phone adoption in the BOP markets in Bangladesh, Pakistan, India, Sri Lanka, Philippines and Thailand.

However, the popular bottom of the pyramid (BOP) proposition has had its fair share of critics too with Karnani (2006) leading the charge by terming it as "…a harmless illusion and potentially a dangerous delusion..." In a series of articles, Karnani (2006) calls the BOP proposition “a mirage.” He avers that, while a few market opportunities do exist, the market at the BOP is generally too small to be profitable for most MNC’s. Instead, he proposes that the private sector can play a key role in poverty alleviation by viewing the poor as producers, and emphasize buying from them, rather than selling to them. In fact, in one of his articles, Karnani (2008) makes a passionate plea to “Help, Don’t Romanticize, The Poor.” Other authors (e.g., Landrum 2007) have also contributed to this growing debate.

In the current paper, we propose to analytically examine the BOP proposition, and recommend whether or not it is an attractive proposition, or, alternatively, the conditions under which it turns out to be a viable proposition. An added contribution of our work is that, as a benchmark, we also consider the case of non-profit organizations (NPO) in our analysis, which will add significant value to resolving the debate about the BOP markets.

2.2 Non-Profit Organizations and Marketing

A nonprofit organization is formed with the purpose of maximization of social benefit rather than the maximization of profit for owners or investors. By definition itself, there are some significant differences between a for-profit organization (FPO) and a non-profit organizations (NPO) (Weinberg 1983). Since the objective function is not profit maximization, it is some other function of demand such as usage maximization. Further, few NPOs meet all their costs from user fees alone. This implies that they must design their marketing programs which appeal and provide benefits to both users and funding sources, such as donors, taxpayers, or government. It is clear that the modeling approaches to design marketing mix programs for NPO’s end up giving very different results which one would intuitively expect in the case of FPO’s (refer Weinberg 1983 for an interesting contrast). It is also clear that, in the context of BOP markets considered in this paper, the non-profit organizations are an apt benchmark to compare with because of the consonance of the objective function of NPO’s with the setting of BOP markets. We now review some relevant research work in the area of marketing in the NPO sector.

The interest in marketing in nonprofit sector began with several influential papers and books such as Kotler and Zaltman (1971), Kotler (1979), Kotler and Fox

(1985), Kotler and Andreasen (1987), Lovelock and Weinberg (1990), Goldberg et al. (1997), and Sargeant (1999a). A comprehensive review of nonprofit marketing landscape is given by Bennett and Sargeant (2005).

Bennett and Kottasz (2000) discuss the advertising style and the recruitment of charity volunteers by the nonprofits. Referring to the important issue of branding by nonprofits, Ritchie et al. (1998) mention that branding facilitates the development of trust between the nonprofit and its constituencies, provide insulation from competitive pressures, and raise the organization’s profile. However, they also mention that brands are not appropriate for all nonprofit organizations, and the decision to adopt a branding strategy is one that requires careful consideration. Caldwell (2000) addresses the research issues related with the emergence of museum brands. Quelch et al. (2004) also make a strong case for ‘mining’ gold in not-for-profit brands. Sargeant (1999b, 2001) addresses the important issues of donor behavior and lifetime value for fundraising strategies. Weinberg and Ritchie (1999), and Liu and Weinberg (2004) address the important issues of competition in the context of nonprofits.

2.3 Modeling of Innovation Diffusion and Optimal Control Approach

Starting with the founding models of Fourt and Woodlock (1960), Mansfield (1961), Bass (1969), and Fisher and Pry (1971), research on modeling of the diffusion of innovations in marketing has resulted in an extensive literature (Mahajan et al., 1990). Product price and advertising have played an important role in the diffusion process. Robinson and Lakhani (1975) introduced price in a multiplicative form. An alternative formulation was introduced by Mahajan and Peterson (1978), where the potential population was considered as a function of price. Parker and Sarvary (1997) present an approach for formulating dynamic pricing and advertising strategies using an approach labeled as decision calculus. Teng and Thompson (1983) have dealt with the optimal pricing and quality policies for the introduction of a new product.

Several authors have used optimal control theory in diffusion of innovation framework to derive normative optimal policies (refer Sethi and Thompson, 2000; Kamien and Schwartz, 1981). Feichtinger et al. (1994) provide an excellent review of developments in the area of dynamic optimal control models in advertising. Simon (1982) proposed ADPULS model to deal with the issue of advertising wear-out over time. His results show that a pulsation advertising policy is better than constant spending policy. Horsky and Simon (1983) examine the effect of advertising on the diffusion of telephonic banking. Their normative advertising policy is to advertise heavily when the product is introduced and to reduce the level of advertising as sales increase and the product moves through its life cycle. Mahajan and Muller (1986) deal with policies for awareness generation and compare five different types of advertising policies: blitz, pulsing, chattering, even, and maintenance. Bayus (1994) develops an optimal control model by incorporating the dynamic effects of incremental product improvements and price on product diffusion. The optimal policy results in Bayus's (1994) paper state that if demand is very sensitive to product enhancements, then prices should decline over time.

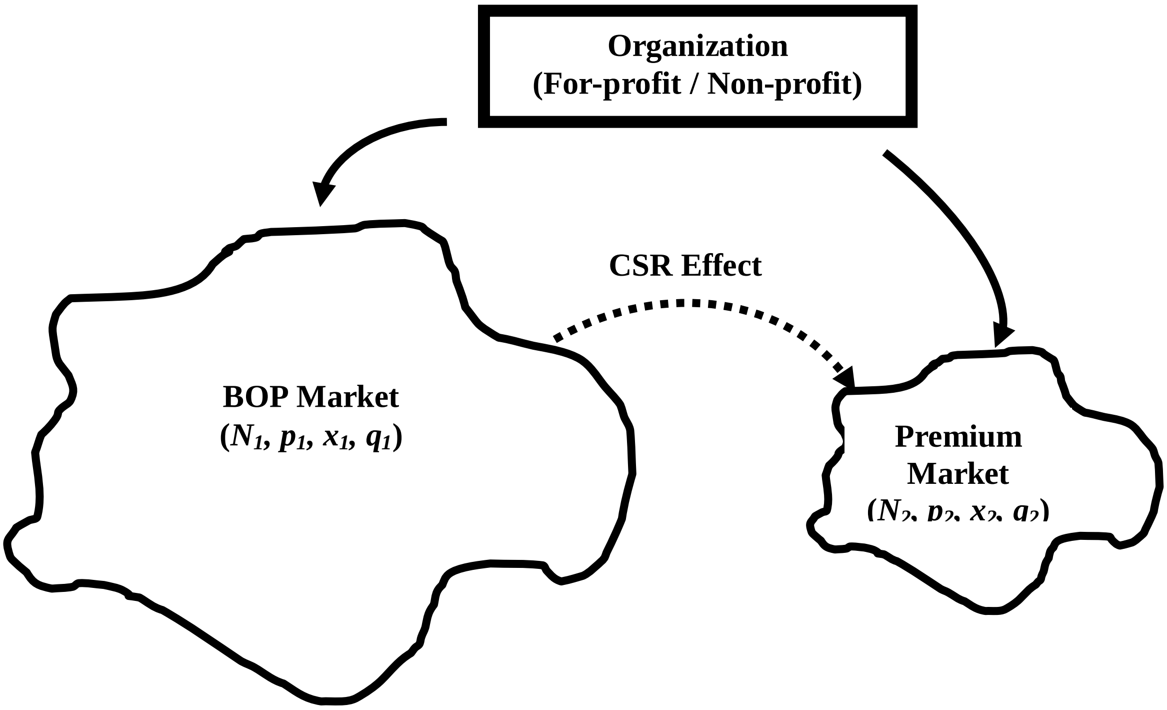
Horsky and Mate (1988) develop dynamic advertising strategies of two competing durable goods producers using stochastic closed-loop control problem. More recently, Buratto et al. (2006) develop optimal policies for advertising a new product in a segmented market. Swami and Khairnar (2006) examine the case of developing optimal normative policies for marketing of products with limited availability. Marinelli (2007) considered the case of stochastic control problems related to optimal advertising. More recently, Swami and Dutta (2010) propose optimal normative advertising strategies for new product diffusion in the context of emerging markets and lead-lag markets scenario. In the current paper, the demand function of our dynamic model is based on the pure innovation model by Fourt and Woodlock (1960).

3 Problem Formulation

We now present the formulation of the problem considered in this paper. The problem considered is of an organization which sells its product/service to two broad markets (refer Figure 1). The first market is the BOP market with a huge market potential, N_1. The second market is the Premium market with much smaller market potential, N_2. Thus, $N_1 >> N_2$.

The organization needs to decide how much price to charge in the two markets, p_1 and p_2, respectively. Since the capacity to pay in the BOP market is much less, it can be expected that $p_2 >> p_1$. However, since the prices are decision variables in the context of our paper, it would be interesting to examine whether $p_2 >> p_1$ indeed turns out to be the case as an outcome of our model results. The organization also needs to decide how much to invest in advertising / promoting the product in the two markets, x_1 and x_2, respectively. Intuitively, it is unclear as to what the directional relation should be between the promotional expenditures in the two markets, which makes it an interesting decision variable to examine.

Figure 1: Organizational Context in a BOP Market



As discussed in the introduction section, the product/service offered to the two markets is differentiated in such a way that the base product (of appropriate quality) is available to the BOP market, while the higher-end product is available to the Premium market. As shown in Figure 1, the two markets are linked to each other such that there is a positive effect of customer base in the BOP market on the diffusion of product in the premium market. This is akin to the outcome of the image-building CSR initiatives of an organization such that the lesser the price it charges in the BOP market (and hence the greater is its demand in that market), the greater is the positive impact on its demand in the premium market. For example, in discussing the payoffs of the distribution strategies for reaching the BOP's, Vachani and Smith (2008, p. 79) mention:

> *Corporate Image*: ITC's innovative use of technology to empower farmers has won considerable recognition. The company has received a number of international awards including the Stockholm Challenge, 2006. In a speech in May 2007, the President of India, Dr. Abdul Kalam, praised ITC's e-Choupal initiative as a sustainable model for raising farmers' incomes and productivity. The Government of India's 2007 annual Economic Survey recognizes ITC's e-Choupal initiative as an example of "novel private sector initiatives to improve the marketing channels in agriculture," which are seen as important for the overall development of India's agricultural sector.

Further, scale of operations is another enabler at the BOP markets. In reference to the importance of scale in BOP markets, Rangan et al. (2011, p. 113) mention:

> Indeed, decent profits can be made at the base of the pyramid if companies link their own financial success with that of their constituencies. In other words, as companies make money, the communities in which they operate must benefit by, for example, acquiring basic services or growing more affluent. This leads to more income and consumption—and triggers more demand within the communities, which in turn allows the companies' businesses to keep growing. *A corollary of that principle is that from the very beginning, scale is critical: Tentative forays into the base of the pyramid do not yield success.*

From the above discussion, it is clear that any modeling attempt to model the BOP market must take its following features in consideration. The first is the *CSR effect*, that is, the effect that the demand in the BOP market has on the demand in the premium market. This effect is termed as CSR effect, as any effort that the organization takes up to serve the BOP market can be expected to create positive brand equity for its product / service in the Premium market. While studies have begun to be conducted on correlation between success at the BOP markets, CSR activities and financial performance of an organization, some early indicators seem to support such a connection (see, for example, Davidson 2009). Indeed, some institutes, like Institute for Innovation and Social Entrepreneurship at ESSEC, France (refer http://www.iies.fr/en/content/corporate-social-responsibility-csr-base-pyramid-bop) appear to use the terms BOP and CSR as synonymous to each other. Other researchers, such as Augustine (2008) advocate creation of a social value index to address the pertinent question: *Do Companies That Engage in BOP Markets Outperform the Market?* Agrawal et al. (2011) also raise a similar concern: *CSR and BOP Marketing: Are they Two Sides of the Same Coin?* Ramani and Mukherjee (2011) find support for the BOP-CSR link through cases from India. In the current paper, we operationalize the CSR effect through the prices charged in the BOP market. Thus, the lower the price charged in the BOP market, the stronger is the CSR effect in the premium market. Conversely, the higher the prices charged in the BOP market, the more negative is the impact on demand in the premium market.

Another effect is the *scale of operations* effect. Since the BOP market is huge, it is natural to expect that the scale effects would dominate while operating in the BOP market (Vachani and Smith 2008). Such is the importance of the scale effect in the BOP market that, even the staunch critic of the BOP proposition, Aneel Karnani, concedes in the paper by Garrette and Karnani (2010, p. 2) that "only…Grameen Bank and Aravind Eye Care…attained a scale sufficient to transform a 'business model' into a 'solution'."[1] In this paper, we operationalize the scale effect by its impact in cost reduction.

Also, it is clear that infrastructural, and other *market development* constraints remain paramount in developing BOP markets (refer Vachani and Smith 2008; Garrette and Karnani 2010). In our model, we operationalize these constraints by incorporating fixed costs of developing the market.

Finally, in our paper, while considering the scenario of BOP markets, we aver that both for-profit and non-profit organizations are equally suited to cater to such markets. In fact, Garrette and Karnani (2010), in quoting the 'only' successful examples of BOP proposition, namely, Grameen Bank and Aravind Eye Care, seem to indicate that perhaps NPO's are better suited to serve these markets. Thus, in order to operationalize the case of NPO's, we also include in our model the effects of *donor subsidy* to the NPO's.

With the above features of the problem, we now explain the model development for BOP markets in the following section. We first discuss the static model, and then the dynamic optimal control model.

[1] This conclusion was based on a study done originally by Monitor Consulting Group (2009).

4 Modeling and Analysis

4.1 Static Model

We first develop a static model for the problem situation described above. As shown in Figure 1, our model consists of a manufacturer, who has to decide on the pricing and advertising levels in the two markets. These are denoted by p_1 and x_1, respectively, for the BOP market, and p_2 and x_2 for the premium market. Similar to the models proposed in the extant literature (Savaskan and van Wassenhove 2006; Swami and Shah 2012), the demand functions in the two markets are assumed to be linearly downward sloping in price, and upward sloping in advertising (or promotion) efforts. Let the demand functions for the two markets be denoted by q_1 and q_2, respectively.

The following notation is used in the models of this section:

N_1	:	Market potential of BOP market
N_2	:	Market potential of Premium market
γ_1	:	Price sensitivity of BOP market
γ_2	:	Price sensitivity of Premium market
α_1	:	Advertising/Promotion sensitivity of BOP market
α_2	:	Advertising/Promotion sensitivity of Premium market
β_2	:	CSR effect of BOP market in Premium market
p_1	:	Price per unit in BOP market (Decision variable)
p_2	:	Price per unit in Premium market (Decision variable)
x_1	:	Advertising/Promotion level in BOP market (Decision variable)
x_2	:	Advertising/Promotion level in Premium market (Decision variable)
q_1	:	Demand function of BOP market
q_2	:	Demand function of Premium market
h_1	:	Cost function of BOP market
h_2	:	Cost function of Premium market
δ_1	:	Cost parameter of advertising/promotion in BOP market
δ_2	:	Cost parameter of advertising/promotion in Premium market
d_1	:	Cost reducing scale parameter in the BOP market
K_1	:	Fixed cost of operating in BOP market
K_2	:	Fixed cost of operating in Premium market

c_1 : Variable cost per unit in BOP market

c_2 : Variable cost per unit in Premium market

S : Fixed donor subsidy for a non-profit organization

π : Objective (profit) function of a for-profit organization (FPO)

Q : Objective (usage) function of a non-profit organization (NPO)

Then, based on the assumptions of our model, and the above notation, the two demand functions can be shown as follows:

BOP Market: $$q_1 = (N_1 - \gamma_1 p_1 + \alpha_1 x_1) \quad (1)$$

Premium Market: $$q_2 = (N_2 - \gamma_2 p_2 + \alpha_2 x_2 - \beta_2 p_1) \quad (2)$$

Some noteworthy points about the above demand functions are as follows: As mentioned earlier, the demand functions are linearly downward sloping in price, and upward sloping in advertising (or promotion) efforts. However, the demand function for the Premium market is also affected by the prices in the BOP market through the CSR effect, β_2. We now consider the case of a for-profit organization.

4.1.1 For-Profit Organization (FPO)

The objective function for a FPO is the maximization of total profit from the two markets, as given below.

Maximize $$\pi = \{(p_1 - c_1)q_1 - h_1 - \delta_1 x_1^2\} + \{(p_2 - c_2)q_2 - h_2 - \delta_2 x_2^2\} \quad (3)$$

The cost functions are specified as follows:

$$h_1 = K_1 - d_1 q_1^2 \quad (4)$$

$$h_2 = K_2 \quad (5)$$

The following points about the modeling of above profit and cost functions are in order here. First, the modeling of the cost of advertising/promotion efforts in the two markets as $\delta_1 x_1^2$ and $\delta_2 x_2^2$, respectively, accounts for the saturation effects of such expenditure. The specific functional form used in based on the similar forms used for modeling effort variables in the extant literature (see, for example, Savaskan and van Wassenhove 2006; Swami and Shah 2012). Second, the modeling of cost function, $h_1 = K_1 - d_1 q_1^2$, accounts for the scale effects in the BOP market. Thus, cost reduction (or spreading of costs) is achievable at larger volumes in the BOP market. Further, since no such scale effect is envisaged for the Premium market, only fixed cost K_2 is used in its cost function.

Assuming concavity of the objective function, the first-order conditions for π yield the following optimal functional values of p_1, p_2, x_1, and x_2, as shown below (refer Appendix for the complete derivation of results)[2]:

$$p_1^* = c_1 + \frac{q_1^*}{\gamma_1} - d_1 q_1^* - \left(\frac{\beta_2}{\gamma_1\gamma_2}\right) q_2^* \tag{6}$$

$$p_2^* = c_2 + \frac{q_2^*}{\gamma_2} \tag{7}$$

$$x_1^* = \frac{\alpha_1}{2\gamma_1\delta_1}\left\{q_1^* - \left(\frac{\beta_2}{\gamma_2}\right) q_2^*\right\} \tag{8}$$

$$x_2^* = \frac{\alpha_2}{2\gamma_2\delta_2} q_2^* \tag{9}$$

Although the above expressions are not closed-form, since the demand functions q_1 and q_2 are functions of p_1, p_2, x_1, and x_2, some interesting observations can be made from these initial results:

- We first discuss the case of pricing in the Premium market. In this market, as expected, a monopolist cost-plus-markup rule results for the optimal pricing strategy. The price per unit, p_2, comprises of variable cost per unit, c_2, plus some mark-up. This mark-up depends on the demand function responsiveness of the pricing and advertising strategies followed by the FPO in the Premium market. Of course, through the CSR effect, it also depends on the pricing strategy followed by the organization in the BOP market.
- Interestingly, in the case of pricing in the BOP market also, the cost-plus-markup rule prevails, but it is moderated by the effects of scale and CSR. Thus, the price per unit in the BOP market, p_1, comprises of variable cost per unit, c_1, plus some mark-up depending on the demand function responsiveness. However, in the case of BOP market, the cost savings resulting due to the scale effect (d_1) in the BOP market, and the CSR

[2] Our major objective to present these optimal functional forms is to compare the similar expressions for the case of NPO's. The case of closed-form exact solution will be taken up in a subsequent section.

enhancing effect (β_2) of the BOP on Premium market, is "passed" on as reduced prices to the BOP market. This reduction in BOP market prices would have further demand-enhancing effects in both BOP and Premium markets.

- Our optimal advertising expression in the BOP market shows that, as demand in the BOP market increases, there is a tendency to increase the advertising expenditure, x_1. However, this increase is moderated by the demand responsiveness to advertising, price sensitivity of the BOP market, and the cost of advertising. As expected, x_1 increases with increased demand responsiveness to advertising, and decreases with increased price sensitivity and cost of advertising. The decrease in x_1 as a result of increased price sensitivity could be due to the fact that, with increased price sensitivity, demand is more responsive to price changes, and perhaps spending on advertising should be reduced.
- Further, and interestingly, the expression for x_1 shows dependence on the demand in the Premium market also through the CSR effect. Specifically, if CSR effect is significant in the BOP market, then the FPO would reduce the advertising expenditure in the BOP market to reap the benefits of the CSR "externalities" prevalent in the market. The optimal advertising expressions for the premium market are similar to the BOP market except for the CSR effect.

We now consider the case of a NPO (non-profit organization) operating under similar conditions to examine the differences in the pricing and advertising strategies for the two types of organizations: FPO vs. NPO.

4.1.2 Non-Profit Organization (NPO)

The objective function for a NPO is usage maximization, as opposed to profit maximization for a FPO (see, for example, Weinberg 1983). Thus, the objective function for a NPO in this setting is to maximize the demand generated from the two markets.

Thus, its objective function can be written as follows:

Maximize $$Q = q_1 + q_2 \tag{10}$$

However, in the case of a NPO, this objective function has to be subject to a non-profit constraint: *any surplus generated plus donor subsidy should be sufficient to cover the costs of operations.* In the context of our model, this non-profit constraint can be shown as below:

$$(p_1 - c_1)q_1 + (p_2 - c_2)q_2 + S \geq h_1 + h_2 + \delta_1 x_1^2 + \delta_2 x_2^2 \quad (11)$$

To find out the optimal functional values of p_1, p_2, x_1, and x_2, in the case of a NPO, we need to form a Lagrangean function, as shown below. Here, λ (≥ 0) is the Lagrange multiplier.

$$L = q_1 + q_2 + \lambda * \{(p_1 - c_1)q_1 + (p_2 - c_2)q_2 + S - h_1 - h_2 - \delta_1 x_1^2 - \delta_2 x_2^2\}$$

(12)

The first-order conditions for L with respect to p_1, p_2, x_1, and x_2, and a Lagrange condition (13), yield the optimal functional values for the four decision variables. The Lagrangean condition (based on Karush-Kuhn-Tucker conditions) is as follows:

$$\lambda * \{(p_1 - c_1)q_1 + (p_2 - c_2)q_2 + S - h_1 - h_2 - \delta_1 x_1^2 - \delta_2 x_2^2\} = 0 \quad (13)$$

Equation (13) can be interpreted as follows: The expression which is multiplied with λ is the surplus that the NPO generates after recovering all its costs. However, by definition of non-profits, since the NPO would like to reduce this surplus as much as possible, it would be desirable if this surplus is zero, or close to zero. The condition that would ensure this is $\lambda > 0$, that is, λ being strictly positive. Thus, in the NPO problem, we need to find optimal expressions of p_1, p_2, x_1, and x_2 in terms of λ. Then, if we put those expressions in the following condition of surplus = 0, we can check whether $\lambda > 0$.

$$\{(p_1 - c_1)q_1 + (p_2 - c_2)q_2 + S - h_1 - h_2 - \delta_1 x_1^2 - \delta_2 x_2^2\} = 0 \quad (14)$$

The Karush-Kuhn-Tucker (KKT) conditions for the function L give the following optimal values:

$$p_{1,NPO}^* = c_1 + \frac{q_1^*}{\gamma_1} - d_1 q_1^* - \left(\frac{\beta_2}{\gamma_1 \gamma_2}\right) q_2^* - \frac{1}{\lambda} \quad (15)$$

$$p^*_{2,NPO} = c_2 + \frac{q^*_2}{\gamma_2} - \frac{1}{\lambda} \tag{16}$$

$$x^*_{1,NPO} = \frac{\alpha_1}{2\gamma_1\delta_1}\left\{q^*_1 - (\frac{\beta_2}{\gamma_2})q^*_2\right\} \tag{17}$$

$$x^*_{2,NPO} = \frac{\alpha_2}{2\gamma_2\delta_2} q^*_2 \tag{18}$$

The above expression appear quite similar to those for the FPO case, except that, in the case of the pricing strategy, there is an additional (-1/λ) term. Using the expressions from Equations (15) to (18), and putting them in (14), it can be shown that (refer Appendix for details):

$$\lambda = \frac{q_1 + q_2}{q_1^2\Delta_1 + q_2^2\Delta_2 + q_1 q_2\Delta_3 + S - K_1 - K_2} \tag{19}$$

Where

Δ_1=$(\frac{1}{\gamma_1} - d_1 - \frac{\alpha_1^2}{4\delta_1\gamma_1^2})$,

Δ_2=$(\frac{1}{\gamma_2} - \frac{\alpha_1^2\beta_2^2}{4\delta_1\gamma_1^2\gamma_2^2})$,

Δ_3=$(\frac{\alpha_1^2\beta_2}{2\delta_1\gamma_1^2\gamma_2} - \frac{\beta_2}{\gamma_1\gamma_2})$

Thus, if it could be ensured, that

(i) $\Delta_1, \Delta_2, \Delta_3 > 0$, and

(ii) $S - K_1 - K_2 > 0$

then λ > 0.

Condition (i) a set of regularity conditions on the problem parameters, while Condition (ii) specifies that the donor subsidy should be strictly greater than the sum of the fixed costs in the two markets.

With λ > 0, we have the following proposition:

Proposition 1: *In the case of dealing with BOP markets, under the conditions (i) and (ii), we have the following results:*

a) *A non-profit organization (NPO) charges lesser price per unit in both BOP and Premium markets, as compared to the for-profit organization (FPO).*

b) *A non-profit organization (NPO) spends equal amount of money in advertising or promoting the product/service as that spent by a for-profit organization (FPO).*

Some observations on Proposition 1 are provided below:

- Part (a) of the proposition finds support in the non-profit literature (e.g., refer Weinberg 1983; Liu and Weinberg 2004). However, this result also depends on the assumption of donor subsidy being sufficiently large to cover certain costs for the non-profits.
- In the above model, we have used the case of a fixed donor subsidy, S. However, S itself can be made of the demand functions, particularly q_1 in the BOP market. Our analysis shows that it will not affect the directionality of our results.
- Part (b) of the proposition suggests that when it comes to promoting the product / service, the NPO's may well promote the product/service like a for-profit organization. Indeed, Weinberg (1983) shows that an NPO may promote the product/service at an even higher level than that by a FPO.[3] However, the caveats suggested in Ritchie et al. (1998) must be kept in mind while promoting or building non-profit brand. Thus, while the extent of promotion might be similar to that of a FPO, the manner in which the promotion is done need not be.

Having compared the cases of FPO vs. NPO in a bottom-of-pyramid market, we now turn our attention to a specific case of a FPO to find closed form solutions of prices and advertising in the BOP and Premium markets.

4.1.3 A Specific Functional Form and an Exact Solution for a For-Profit Organization (FPO) in BOP Market

In order to come up with an exact solution for prices and advertising, we need to simplify the model presented in Equations (1) to (5) by using slightly more parsimonious functional forms. These are revised as given below:

[3] In case of Weinberg (1983), donor subsidy was a function of demand generation.

Demand functions

BOP Market: $q_1 = (N_1 - p_1 + \alpha_1 x_1)$ (19)

Premium Market: $q_2 = (N_2 - p_2 + \alpha_2 x_2 - \gamma p_1)$ (20)

Objective function

Maximize $\pi = \{(p_1 - c_1)q_1 - h_1 - \delta_1 x_1^2\} + \{(p_2 - c_2)q_2 - h_2 - \delta_2 x_2^2\}$

(21)

Cost functions

$h_1 = K_1 - d_1 q_1$ (22)

$h_2 = K_2$ (23)

Specifically, the changes made in the model presented by Equations (19) to (23) can be summarized as follows: (i) The parameters for the price sensitivity have been set equal to 1, (ii) The CSR effect is labeled as γ, and (iii) The cost-reducing scale effect has been linearized, as shown in (23). It is clear that, even in this parsimonious model, the essential features of the problem formulation, namely, *two markets* (BOP vs. Premium), *CSR* effect, and *scale* effect, have been maintained. The following optimal values of p_1, p_2, x_1, and x_2 are obtained for this model (refer Appendix for details):

$$p_1^* = c_1 - d_1 + \frac{\gamma(N_2 - c_2) - \gamma^2(c_1 - d_1) - (N_1 - c_1 + d_1)(2 - \Delta_2)}{\gamma^2 - (2 - \Delta_1)(2 - \Delta_2)}$$

(24)

$$p_2^* = c_2 + \frac{\gamma(N_1 - c_1 + d_1) - \{N_2 - c_2 - \gamma(c_1 - d_1)\}(2 - \Delta_2)}{\gamma^2 - (2 - \Delta_1)(2 - \Delta_2)}$$

(25)

$$x_1^* = \frac{\alpha_1}{2\delta_1}\left\{\frac{\gamma(N_2 - c_2) - \gamma^2(c_1 - d_1) - (N_1 - c_1 + d_1)(2 - \Delta_2)}{\gamma^2 - (2 - \Delta_1)(2 - \Delta_2)}\right\}$$

(26)

$$x_2^* = \frac{\alpha_2}{2\delta_2}\left\{\frac{\gamma(N_1 - c_1 + d_1) - \{N_2 - c_2 - \gamma(c_1 - d_1)\}(2 - \Delta_2)}{\gamma^2 - (2 - \Delta_1)(2 - \Delta_2)}\right\}$$

(27)

Where

$\Delta_1 = \frac{\alpha_1^2}{2\delta_1}$, and $\Delta_2 = \frac{\alpha_2^2}{2\delta_2}$

On the basis of the above expressions, we provide the following proposition:

Proposition 2: *In the case of dealing with BOP markets, under the condition that* $\{\gamma^2 - (2-\Delta_1)(2-\Delta_2)\} > 0,$ *we have the following results for a for-profit organization (FPO):*

a) It charges lesser price per unit in the BOP markets as compared to the Premium market

b) It received lesser contribution margin per unit in the BOP market (m_1^*) as compared to the Premium market (m_2^*).

c) The ratio of advertising/promotion done in BOP market to that in the Premium market is given by the following expression:

$$\frac{x_1^*}{x_2^*} = \left(\frac{\alpha_1/\alpha_2}{\delta_1/\delta_2} \right) * \left(\frac{m_1^* + d_1}{m_2^*} \right)$$

Proof of Proposition 2:

(a) From (24) and (25) given above, consider the following difference:

$$p_2^* - p_1^* = c_2 - c_1 + d_1 + \frac{\gamma(N_1 - c_1 + d_1) - \{N_2 - c_2 - \gamma(c_1 - d_1)\}(2-\Delta_2) - \gamma(N_2 - c_2) + \gamma^2(c_1 - d_1) + (N_1 - c_1 + d_1)(2-\Delta_2)}{\gamma^2 - (2-\Delta_1)(2-\Delta_2)} \tag{28}$$

Assuming $\gamma^2 - (2-\Delta_1)(2-\Delta_2) > 0$, the above equation (28) can be written as

$$p_2^* - p_1^* = c_2 - c_1 + d_1 + \frac{\gamma(N_1 - c_1 + d_1 - N_2 + c_2) + \{N_1 - c_1 + d_1 - N_2 + c_2 - \gamma(c_1 - d_1)\}(2-\Delta_2) + \gamma^2(c_1 - d_1)}{\gamma^2 - (2-\Delta_1)(2-\Delta_2)} \tag{29}$$

In (29), $(c_2 - c_1) > 0$, as the variable cost per unit of serving a customer in the Premium market would be significantly greater than that of serving a BOP customer. Since d_1>0, we need to check whether the denominator in the fractional term is positive or not. Since the first two terms are dominated by the term $(N_1 - N_2)$, and since the market potential of the BOP market (N_1) is far greater than that of the Premium market (N_2), it can be said that $(p_2^* - p_1^*) > 0$.

(b) From (24) and (25), consider the following difference:

$$m_2^* - m_1^* = (p_2^* - c_2) - (p_1^* - c_1)$$
$$= d_1 + \frac{\gamma(N_1 - c_1 + d_1) - \{N_2 - c_2 - \gamma(c_1 - d_1)\}(2 - \Delta_2) - \gamma(N_2 - c_2) + \gamma^2(c_1 - d_1) + (N_1 - c_1 + d_1)(2 - \Delta_2)}{\gamma^2 - (2 - \Delta_1)(2 - \Delta_2)} \quad (30)$$

In part (a), we have already proved that this difference is strictly positive. Therefore, $(m_2^* - m_1^*) > 0$.

(c) From (26) and (27), the ratio of advertising/promotion done in BOP market to that in the Premium market can be written as follows:

$$\frac{x_1^*}{x_2^*} = \left(\frac{\alpha_1/\alpha_2}{\delta_1/\delta_2}\right) * \left(\frac{\gamma(N_2 - c_2) - \gamma^2(c_1 - d_1) - (N_1 - c_1 + d_1)(2 - \Delta_2) \Big/ \gamma^2 - (2 - \Delta_1)(2 - \Delta_2)}{\gamma(N_1 - c_1 + d_1) - \{N_2 - c_2 - \gamma(c_1 - d_1)\}(2 - \Delta_2) \Big/ \gamma^2 - (2 - \Delta_1)(2 - \Delta_2)}\right) \quad (31)$$

Using (24) and (25), the above, in turn, can be written as:

$$\frac{x_1^*}{x_2^*} = \left(\frac{\alpha_1/\alpha_2}{\delta_1/\delta_2}\right) * \left(\frac{p_1^* - c_1 + d_1}{p_2^* - c_2}\right) \quad (32)$$

Thus,

$$\frac{x_1^*}{x_2^*} = \left(\frac{\alpha_1/\alpha_2}{\delta_1/\delta_2}\right) * \left(\frac{m_1^* + d_1}{m_2^*}\right)$$

Some observations on Proposition 2 are provided below:

a) From the expressions above, it should be clear that the price charged in the BOP market is not only less than that in the Premium market, but it is *substantially* lesser. Analytically, this result is because of the difference in variable cost structures of the two markets, the scale effect, and the huge difference in market potential terms of the two markets, N_1 and N_2. However, it is consistent with the extant literature which suggests that the prices, and margins (part b of Proposition 2), in the BOP market have to be substantially less than those in the higher-end markets (Garrette and Karnani 2010). The remarkable point about, however, is that we obtain this result even without imposing any specific upper-bound constraint on p_1 to account for the BOP consumer's limited capacity to pay.[4]

b) The next set of pertinent questions that arise are: *Does the FPO make any money from the BOP market? Does it even recover the variable cost per unit from the BOP market?* While a complete numerical analysis would elucidate these issues better, a preliminary observation of the expression in Equation (24) suggests that perhaps it does not recover its variable cost per unit, at least under some combination of the problem parameters. Of course, it would recover some of this "loss" in the BOP market from the Premium market, primarily due to the prevalence of the CSR effects.

c) Finally, the results regarding the ratio of advertising/promotion done in BOP market to that in the Premium market suggest that this ratio would depend on the interplay of three ratios. The first is the ratio of demand-enhancing effectiveness parameters of advertising in the two markets, $({}^{\alpha_1}/_{\alpha_2})$. The second the ratio of advertising cost parameters, $({}^{\delta_1}/_{\delta_2})$. Last is the ratio of contribution margins per unit (moderated by scale parameters) in the two markets, $(\frac{m_1^*+d_1}{m_2^*})$. Thus, the more effective it is to advertise in the BOP market, relatively, the greater will be advertising expenditure in the BOP market. Similarly, the less expensive it is to advertise in the BOP market, relatively,

[4] We have analyzed our model with a specific upper-bound constraint on the prices in the BOP market. However, our results are not materially affected by this restriction. Therefore, we proceed with the parsimonious model of this section.

the greater will be advertising expenditure in the BOP market. However, since the margins in the BOP market are going to be significantly less than their Premium market counterparts, it is much less likely that the FPO would ever advertise or promote the product/service more heavily in the BOP market.

We now discuss a dynamic model, based on optimal control theory, which would help us in arriving at time-based optimal policies for pricing and advertising decisions by the two types of organizations in the BOP market considered in this paper, namely, a for-profit organization (FPO) and a non-profit organization (NPO).

4.2 Dynamic Optimal Control Model

For the dynamic optimal control models, we separate out the decisions of pricing and advertising by the FPO and NPO. We now present the modeling and analysis of the pricing policy.

4.2.1 Dynamic Pricing Policy

We consider a finite-horizon problem. The following notation is used in the pricing policy model:

N_1 : Market potential of BOP market
N_2 : Market potential of Premium market
T : Finite time horizon
γ_1 : Price sensitivity of BOP market
γ_2 : Price sensitivity of Premium market
β_2 : CSR effect of BOP market in Premium market
p_{1t} : Price per unit in BOP market in time period t
p_{2t} : Price per unit in Premium market in time period t
q_{1t} : Cumulative sales in the BOP market till time-period t
q_{2t} : Cumulative sales in the Premium market till time-period t
q'_{1t} : Instantaneous (per-period) sales in BOP market in time-period t (dq_1/dt)[5]
q'_{2t} : Instantaneous (per-period) sales in Premium market in time-period t (dq_2/dt)
h_1 : Cost per unit (fixed) in BOP market

[5] The x' notation is meant to indicate derivative with respect to t. Thus, $x' = dx/dt$

h_2 : Cost per unit (fixed) in Premium market

π : Objective (profit) function of a for-profit organization (FPO)

We first consider the case of a for-profit organization.

Pricing Policy of an FPO

The following optimal control model is considered in this case.[6]

Objective Function

Maximize

$$\pi = \{(p_1 - h_1)q_1' + (p_2 - h_2)q_2'\}dt \tag{33}$$

Subject to Constraints

$$q_1' = (N_1 - q_1) * (\alpha_1 - \gamma_1 p_1) \tag{34}$$

$$q_2' = (N_2 - q_2) * (\alpha_2 - \gamma_2 p_2 + \beta_2 q_1) \tag{35}$$

$$p_2 \geq p_1 \tag{36}$$

$$0 \leq q_1 \leq N_1, \quad 0 \leq q_2 \leq N_2 \tag{37}$$

As shown in the above model, (33) represents the cumulative profits from the two markets over T time periods. Equations (34) and (35) represent the product adoption/diffusion equations of demand in the two markets following Fourt and Woodlock (1960) model's pure innovation effects and an explicit expression for price sensitivity in the two markets.[7] The CSR effect is included here in (35) by the parameter β_2.[8] Equation (35) explicitly specifies that the prices in the Premium market are bounded on the lower side by the prices in the BOP market. Finally, (37) specifies the end conditions using market potentials on the cumulative adoptions in the two markets.

The above model is a dynamic optimization problem in optimal control theory framework. We use the Pontryagin's maximum principle to characterize the optimal

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[6] The subscript *t* is dropped for better exposition from this point onwards.

[7] Pure innovation model is the most parsimonious form at this initial level of specification of the model. This model can readily be extended to include imitation effects using Bass (1969) model, or its later variants to include other marketing variables.

[8] Note that in the optimal control model, we consider the complete effect of "stock" of consumers in BOP market on the adoptions in the Premium market. Although this has been done primarily for the parsimony of the model concerned, it can be reasonably interpreted as the "influence" that the adopters in the BOP market would have on the diffusion in the Premium market.

policy. We include constraint into the objective function, by multiplying it with the shadow prices, λ_1 and λ_2, to form the Hamiltonian function given below:

$$H = \{(p_1 - h_1 + \lambda_1)q_1' + (p_2 - h_2 + \lambda_2)q_2'\} \tag{38}$$

Then, to include the constraint (36) explicitly, we form a Lagrangean function as given below:

$$L = \{(p_1 - h_1 + \lambda_1)q_1' + (p_2 - h_2 + \lambda_2)q_2'\} + \mu(p_2 - p_1) \tag{39}$$

We assume that the optimal solution exists at every point in time, t, and therefore the derivative of the Hamiltonian with respect to p_1 and p_2 must vanish on the optimal pricing path $p*$. Thus, the following optimality conditions result:

$$(i)\frac{\partial L}{\partial p_1} = 0, \quad (ii)\frac{\partial L}{\partial p_2} = 0\,, \quad (iii)\,\mu \geq 0,$$
$$(iv)\mu(p_2 - p_1) = 0, \quad (v)\lambda_1' = -L_{q_1}, \quad (vi)\,\lambda_2' = -L_{q_2}$$

The maximum principle states that the optimal solutions, $p_1*(t)$, and $p_2*(t)$, to system of Equations (33) to (37), has to maximize the Hamiltonian, H (and Lagrangean L), at each instant t, with the sales $q_1(t)$, and $q_2(t)$, and the shadow prices $\lambda_1(t)$, and $\lambda_2(t)$ following the sales equations stated in Equations (34) and (35), as well as the differential equations given in conditions (v) and (vi) above. On the basis of the above conditions, the following expressions result for p_1' and p_2':

$$2p_1' = \left\{-\frac{{q_1'}^2}{\gamma_1(N_1-q_1)^2} + \frac{\beta_2}{\gamma_2}q_2'\right\} \tag{40}$$

$$2p_2' = \left\{-\frac{{q_2'}^2}{(N_2-q_2)^2} + \frac{\beta_2}{\gamma_2}q_1'\right\} \tag{41}$$

On the basis of the above expressions, the following proposition is obtained:

Proposition 3: *The following pricing policy results for a for-profit organization (FPO):*

a) ***BOP Market****: The pricing strategy depends on a ratio of instantaneous sales in the BOP markets as compared to the Premium market in the following way:*

- *If* $\left\{\left(\frac{{q_1'}^2}{q_2'}\right) < \frac{\beta_2\gamma_1(N_1-q_1)^2}{\gamma_2}\right\}$*, then price increases over time.*
- *If* $\left\{\left(\frac{{q_1'}^2}{q_2'}\right) > \frac{\beta_2\gamma_1(N_1-q_1)^2}{\gamma_2}\right\}$*, then price decreases over time.*
- *If* $\left\{\left(\frac{{q_1'}^2}{q_2'}\right) = \frac{\beta_2\gamma_1(N_1-q_1)^2}{\gamma_2}\right\}$*, then price remains stable over time.*

b) ***Premium Market****: Price gradually increases over time.*

Proof of Proposition 3:

(a) From (40),

$$2p_1' = \left\{-\frac{{q_1'}^2}{\gamma_1(N_1-q_1)^2} + \frac{\beta_2}{\gamma_2}q_2'\right\}$$

Now, $p_1' > 0$, if $\left\{-\frac{{q_1'}^2}{\gamma_1(N_1-q_1)^2} + \frac{\beta_2}{\gamma_2}q_2'\right\} > 0$.

Rearranging the terms, we get the condition

$\left\{\left(\frac{{q_1'}^2}{q_2'}\right) < \frac{\beta_2\gamma_1(N_1-q_1)^2}{\gamma_2}\right\}$, for which $p_1' > 0$, or the price will increase over time in the BOP market. The other two conditions can be derived similarly.

Since initially sales would rise only gradually in the BOP market, and would not be able to dominate over the term $(N_1 - q_1)^2$, it can be expected that the prices will increase initially, even if for a short period. This will happen till the equality is reached when prices remain stable and then drop afterwards.

(b) From (41),

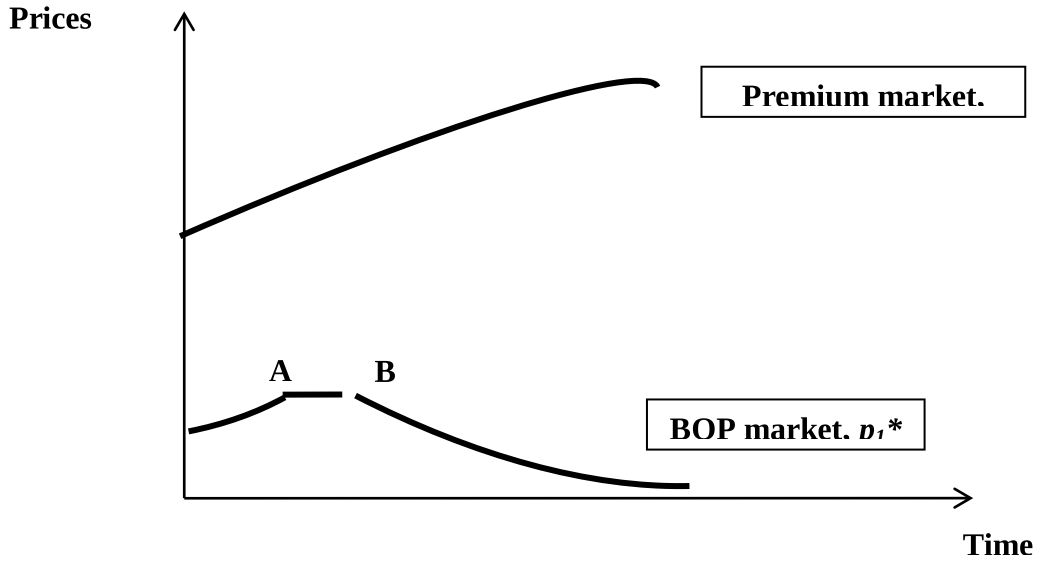
$$2p_2' = \left\{-\frac{{q_2'}^2}{(N_2-q_2)^2} + \frac{\beta_2}{\gamma_2}q_1'\right\}$$

$p_2' > 0$, if $\left\{-\frac{{q_2'}^2}{(N_2-q_2)^2} + \frac{\beta_2}{\gamma_2}q_1'\right\} > 0$, or

$$q_1' > \frac{\gamma_2{q_2'}^2}{\beta_2(N_2-q_2)}$$

Since the instantaneous sales in the BOP market are likely to dominate that of the Premium market, it can be inferred that the above inequality will hold throughout the planning horizon. Thus, $p_2' > 0$, or price would gradually increases over time in the Premium market. Based on the above results, the following pricing policy patterns can be anticipated for the two markets by a FPO (refer Figure 2):

Figure 2: Pricing Policy for the BOP and Premium Markets by a For-Profit Organization (FPO)



As shown in Figure 2, prices in the Premium market are always greater than those in the BOP market. Further, prices in the Premium market increase over time, as the product becomes more popular. The increase in CSR effect also contributes to increase in the prices in the Premium market. These results are consistent with the previous literature (Bayus 1994; Swami and Khairnar 2006).

As far as BOP market is concerned, prices first increase, then remain stable, and then fall steadily. Point A is the threshold up to which the prices will increase. This corresponds to the first condition mentioned in Proposition 3 (a). Point B is the threshold after which prices will start decreasing in the BOP market. This point corresponds to the second condition mentioned in Proposition 3 (a). Between these two points, the last condition of Proposition 3(a) prevails, and the prices remain stable. The increase in prices in BOP market is somewhat surprising. However, we recognize that the optimal policy results only suggest the general direction of the policy, not how long the increase would be in terms of duration. Looking at the threshold condition itself, one could observe that Point A could be reached faster under the following conditions: (i) if sales in BOP market increase rapidly, (ii) if the CSR effect is weak, and / or (iii) if the price sensitivity of the Premium market increases with respect to the BOP market. While the first two conditions are intuitively appealing, the last condition needs some explanation. If the price sensitivity of the Premium market also increases, in addition to an already price-sensitive BOP market, then the only way for a profit-maximizing monopolist to survive is to reduce prices even in the BOP market. Also, one must recognize that we have not accounted for the scale effects in this model. Scale effects would only help reach Point A faster, and thus the prices would be gradually be reducing in the BOP market.

Pricing Policy of a NPO

The dynamic pricing policy results for a non-profit organization (NPO) are similar to their static counterpart, that is, the *NPO charges lesser price per unit in both BOP and Premium markets, as compared to the for-profit organization (FPO).* In fact, the model developed shows that, over a period of time, the NPO charges close to the per-unit cost, and therefore, passes the surplus benefit to the consumer in both BOP and Premium markets. The details of the results are shown in the Appendix.

4.2.1 Dynamic Advertising Policy

Similar to the case of optimal pricing policy, we consider a finite-horizon problem in the case of adverting policy. The following notation is used in the advertising policy model:

N_1	:	Market potential of BOP market
N_2	:	Market potential of Premium market
T	:	Finite time horizon
α_1	:	Advertising effectiveness in the BOP market
α_2	:	Advertising effectiveness in the Premium market
β_2	:	CSR effect of BOP market in Premium market
x_{1t}	:	Advertising expenditure in the BOP market in time period t
x_{2t}	:	Advertising expenditure in the Premium market in time period t
X	:	Fixed budget available for advertising in any time period
q_{1t}	:	Cumulative sales in the BOP market till time-period t
q_{2t}	:	Cumulative sales in the Premium market till time-period t
q'_{1t}	:	Instantaneous (per-period) sales in BOP market in time-period t (dq_1/dt)
q'_{2t}	:	Instantaneous (per-period) sales in Premium market in time-period t (dq_2/dt)
m_1	:	Contribution margin per unit (fixed) in the BOP market

m_2 : Contribution margin per unit (fixed) in the Premium market

π : Objective (profit) function of a for-profit organization (FPO)

We first consider the case of a for-profit orgnaisation.

Advertising Policy of an FPO

The following optimal control model is considered in this case.

Objective Function

Maximize

$$\pi = \{m_1 q_1' + m_2 q_2' - x_1 - x_2\}dt \tag{42}$$

Subject to Constraints

$$q_1' = (N_1 - q_1) * (\alpha_1 ln\, x_1) \tag{43}$$

$$q_2' = (N_2 - q_2) * (\alpha_2 ln\, x_2 + \beta_2 q_1) \tag{44}$$

$$x_1 + x_2 \leq X \tag{45}$$

$$0 \leq q_1 \leq N_1, \quad 0 \leq q_2 \leq N_2 \tag{46}$$

As shown in the above model, (42) represents the cumulative profits from the two markets over T time periods. Since this is a dynamic model focused on advertising, we have considered fixed contribution margins, m_1 and m_2, for the BOP and Premium markets, respectively. In the profit equation, we treat x_1 and x_2 as fixed expenditure of advertising per period, and directly deduct them from revenues. This is done to ensure consistency with the advertising effectiveness function of natural logarithm (ln x_1 and ln x_2) used in the two sales equations. Previous researchers have also used similar functional forms for advertising effectiveness (Swami and Khairnar 2006).

Equations (43) and (44) represent the product adoption/diffusion equations of demand in the two markets following Fourt and Woodlock (1960) model's pure innovation effects and an explicit expression for advertising effectiveness in the two markets. The CSR effect is included here also in (44) by the parameter β_2. Equation (45) is an advertising budget constraint which limits the total advertising done in the two markets in any time period. (46) specifies the end conditions using market potentials on the cumulative adoptions in the two markets.

The above model is a dynamic optimization problem in optimal control theory framework. We use the Pontryagin's maximum principle to characterize the optimal policy. We include constraint into the objective function, by multiplying it with the shadow prices, λ_1 and λ_2, to form the Hamiltonian function given below:

$$H = \{(m_1 + \lambda_1)q_1' + (m_2 + \lambda_2)q_2' - x_1 - x_2\} \tag{47}$$

Then, to include the constraint (45) explicitly, we form a Lagrangean function as given below:

$$L = \{(m_1 + \lambda_1)q_1' + (m_2 + \lambda_2)q_2' - x_1 - x_2\} - \mu(x_1 + x_2 - X) \tag{48}$$

The following optimality conditions are used in this case:

$$(i)\frac{\partial L}{\partial x_1} = 0, \quad (ii)\frac{\partial L}{\partial x_2} = 0\,, \quad (iii)\ \mu \geq 0,$$

$$(iv)\mu(x_1 + x_2 - X) = 0, \quad (v)\lambda_1' = -L_{x_1}, \quad (vi)\ \lambda_2' = -L_{x_2}$$

On the basis of the above conditions, the following expressions result for x_1' and x_2':

$$x_1' = \left\{-\frac{\alpha_1(N_1-q_1)(N_2-q_2)(m_2+\lambda_2)}{(1+\mu)}\right\} \tag{49}$$

$$x_2' = 0 \tag{50}$$

Since all of the terms (49) are non-negative, and with a separate negative sign, we conclude that $x_1' < 0$. From this inference, and Equation (50), we arrive at the following proposition.

Proposition 4: *The following advertising / promotion policy results for a for-profit organization (FPO):*

a) ***BOP Market****: Advertising efforts should gradually decrease over time.*

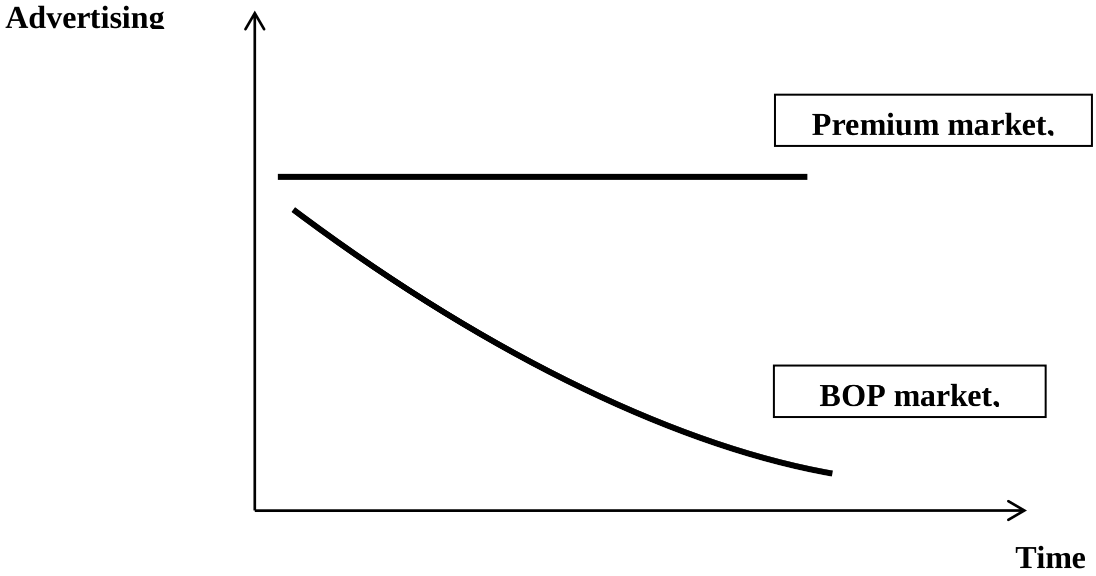
b) ***Premium Market****: Advertising efforts should be stable over time.*

Advertising Policy of a NPO

The dynamic advertising policy results for a non-profit organization (NPO) are similar to their static counterpart, that is, the *non-profit organization (NPO) spends* *equal amount of money in advertising or promoting the product/service as that spent by a for-profit organization (FPO).* Thus, as explained earlier in the context of the static model, when it comes to promoting the product / service, the NPO's may well promote the product/service following a similar advertising/promotion as that followed by a for-profit organization. The details of the results are shown in the Appendix.

Based on the above results, the following advertising policy patterns can be anticipated for the two markets by a FPO (refer Figure 3):

Figure 3: Advertising Policy for the BOP and Premium Markets by a For-Profit Organization (FPO)



5 Conclusions and Directions for Future Research

In this paper, we consider the problem of an organization which sells its product/service to two broad markets. The first market is the bottom-of-pyramid (BOP) market with a huge market potential, but limited ability to pay. The second market is the Premium market with much smaller market potential, but can provide much greater margins to the organization. The organization needs to decide how much price to charge and how much to invest in advertising / promoting the product in the two markets. The product/service offered to the two markets is differentiated in such a way that the base product (of appropriate quality) is available to the BOP market, while the higher-end product is available to the Premium market. The two markets are linked to each other such that there is a positive effect of customer base in the BOP market on the diffusion of product in the premium market. This effect is termed as the *CSR effect*, that is, the effect that the demand in the BOP market has on the demand in the premium market. Another effect is the cost-reducing *scale of operations* effects because of the size of the BOP market. We also consider infrastructural, and *market development* constraints that are operationalized in our model by incorporating fixed costs. Finally, in this paper, while considering the scenario of BOP markets, we posit that both for-profit and non-profit organizations are equally suited to cater to such markets. In the case of NPO's, we also include in our model the effects of *donor subsidy* to the NPO's. We analyze this problem in both static and dynamic contexts.

Our major results show that: (i) A non-profit organization (NPO) charges lesser price per unit in both BOP and Premium markets, as compared to the for-profit organization (FPO), (ii) A non-profit organization (NPO) spends equal amount of money in advertising or promoting the product/service as that spent by a for-profit organization (FPO), (iii) A FPO charges lesser price per unit in the BOP markets as compared to the Premium market, (iv) The FPO receives lesser contribution margin per unit in the BOP market, as compared to the Premium market, (v) For the FPO, the ratio of advertising/promotion done in BOP market to that in the Premium market is governed by the parameters such as relative advertising effectiveness, cost of advertising, and contribution margin per unit in the two markets, (vi) Our dynamic pricing policy results for a FPO show that the prices are gradually increasing in the Premium market, and gradually decreasing in the BOP market, albeit after a threshold level of sales. The dynamic advertising policy results for a FPO show that advertising should gradually be decreased in the BOP market, but should remain stable in the Premium market. The NPO dynamic pricing and advertising results are similar to their static counterparts, though at much lower price levels.

This work can be extended into several directions, which are listed below: (i) We can also model the product quality and R&D effects into this framework, (ii) The consideration of strategic behavior involving multiple manufacturers and retailers would provide additional useful insights, and (iii) The incorporation of uncertainty would provide even richer results in this framework.

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Appendix: Optimal Pricing and Advertising Policies for Bottom of Pyramid Markets: An Analytical Approach

Proposition 1:

FPO Case:

Development of Expressions (6) to (9):

From Equations (3) to (5), we have:

$\pi = \{(p_1 - c_1)q_1 - h_1 - \delta_1 x_1^2\} + \{(p_2 - c_2)q_2 - h_2 - \delta_2 x_2^2\}$

$h_1 = K_1 - d_1 q_1^2, \qquad h_2 = K_2$

Thus, we can write

$\pi = \{(p_1 - c_1)(N_1 - \gamma_1 p_1 + \alpha_1 x_1) - K_1 + d_1 q_1^2 - \delta_1 x_1^2\} + \{(p_2 - c_2)(N_2 - \gamma_2 p_2 +$
$\alpha 2x2 - \beta 2p1) - K2 - \delta 2x22$

(A.1)

The first-order conditions are as follows:

$\frac{\partial \pi}{\partial p_1} = 0,\ \frac{\partial \pi}{\partial p_2} = 0,\ \frac{\partial \pi}{\partial x_1} = 0,\ \frac{\partial \pi}{\partial x_2} = 0$

These conditions give the following equations:

$$\gamma_1 p_1 + \beta_2 p_2 = q_1(1 - 2d_1\gamma_1) + \beta_2 c_2 + \gamma_1 c_1 \quad \text{(A.2)}$$

$$p_2 = c_2 + q_2/\gamma_2 \quad \text{(A.3)}$$

$$(p_1 - c_1)\alpha_1 + 2\alpha_1 d_1 q_1 - 2\delta_1 x_1 = 0 \quad \text{(A.4)}$$

$$(p_2 - c_2)\alpha_2 - 2\delta_2 x_2 = 0 \quad \text{(A.5)}$$

Solving (A.2) to (A.5) gives required expressions in Equations (6) to (9).

NPO Case:

From (12), we have,

$$L = q_1 + q_2 + \lambda * \{(p_1 - c_1)q_1 + (p_2 - c_2)q_2 + S - h_1 - h_2 - \delta_1 x_1^2 - \delta_2 x_2^2\}$$

(A.6)

The first-order conditions give the following equations:

$$\gamma_1 p_1 + \beta_2 p_2 = q_1(1 - 2d_1\gamma_1) + \beta_2 c_2 + \gamma_1 c_1 - \left(\frac{\beta_1 + \gamma_1}{\lambda}\right) \quad \text{(A.7)}$$

$$p_2 = c_2 + \frac{q_2}{\gamma_2} - \left(\frac{1}{\lambda}\right) \quad \text{(A.8)}$$

$$(p_1 - c_1)\alpha_1 + 2\alpha_1 d_1 q_1 - 2\delta_1 x_1 + (\tfrac{\alpha_1}{\lambda}) = 0 \tag{A.9}$$

$$(p_2 - c_2)\alpha_2 - 2\delta_2 x_2 + (\tfrac{\alpha_2}{\lambda}) = 0 \tag{A.10}$$

Solving (A.7) to (A.10) gives required expressions in Equations (15) to (18).

From Equation (14), we have

$$\{(p_1 - c_1)q_1 + (p_2 - c_2)q_2 + S - h_1 - h_2 - \delta_1 x_1^2 - \delta_2 x_2^2\} = 0 \tag{A.11}$$

Putting the values from Equations (15) to (18) in (A.11), we get:

$$q_1^*\left\{\frac{q_1^*}{\gamma_1} - d_1 q_1^* - \left(\frac{\beta_2}{\gamma_1\gamma_2}\right) q_2^* - \frac{1}{\lambda}\right\} + q_2^*\left\{\frac{q_2^*}{\gamma_2} - \frac{1}{\lambda}\right\} + S - K_1 - K_2 + d_1 q_1^{*2}$$

$$-\delta_1\left[\frac{\alpha_1}{2\gamma_1\delta_1}\left\{q_1^* - (\frac{\beta_2}{\gamma_2})q_2^*\right\}\right]^2 - \ \delta_2\left[\frac{\alpha_2}{2\gamma_2\delta_2} q_2^*\right]^2 = 0$$

This can be simplified to

$$q_1^{*2}\left(\frac{1}{\gamma_1} - d_1 - \frac{\alpha_1^2}{4\delta_1\gamma_1^2}\right) + q_1^{*2}\left(\frac{1}{\gamma_2} - \frac{\alpha_1^2\beta_2^2}{4\delta_1\gamma_1^2\gamma_2^2}\right) + q_1^* q_2^*\left(\frac{\alpha_1^2\beta_2}{2\delta_1\gamma_1^2\gamma_2} - \frac{\beta_2}{\gamma_1\gamma_2}\right) +$$

$$S - K_1 - K_2 - \left(\frac{q_1 + q_2}{\lambda}\right) = 0 \tag{A.12}$$

Solving (A.12) for λ gives the relations in Equation (19).

Proposition 2:

The objective function is as follows

Maximize $\quad \pi = \{(p_1 - c_1)q_1 - h_1 - \delta_1 x_1^2\} + \{(p_2 - c_2)q_2 - h_2 - \delta_2 x_2^2\}$

Along with demand and cost functions given in Equations (19)-(21), we can write:

$$\begin{aligned}\pi = &\left\{(p_1 - c_1)(N_1 - p_1 + \alpha_1 x_1) - K_1 + d_1 q_1 \ - \delta_1 x_1^2\right\} \\ &+ \{(p_2 - c_2)(N_2 - p_2 + \alpha_2 x_2 - \gamma p_1) - K_2 - \delta_2 x_2^2\}\end{aligned} \tag{A.13}$$

The first-order conditions give the following equations:

$$2p_1 + \gamma p_2 - \alpha_1 x_1 = N_1 + c_1 - d_1 + \gamma c_2 \tag{A.14}$$

$$\gamma p_1 + 2p_2 - \alpha_2 x_2 = N_2 + c_2 \tag{A.15}$$

$$\alpha_1 p_1 - 2\delta_1 x_1 = \alpha_1(c_1 - d_1) \tag{A.16}$$

$$(p_2 - c_2)\alpha_2 - 2\delta_2 x_2 = 0 \tag{A.17}$$

Solving (A.14) to (A.17) gives required expressions in Equations (24) to (27).

Proposition 3:

From Equation (39), we have,

$L = \{(p_1 - h_1 + \lambda_1)q_1' + (p_2 - h_2 + \lambda_2)q_2'\} + \mu(p_2 - p_1)$

We use the following optimality conditions:

$(i)\dfrac{\partial L}{\partial p_1} = 0, \quad (ii)\dfrac{\partial L}{\partial p_2} = 0\,, \quad (iii)\ \mu \geq 0,$

$(iv)\mu(p_2 - p_1) = 0, \quad (v)\lambda_1' = -L_{q_1}, \quad (vi)\ \lambda_2' = -L_{q_2}$

Using (i), we have:

$q_1' - \gamma_1(N_1 - q_1)(p_1 - h_1 + \lambda_1) - \mu = 0$

Or we can write

$p_1 = \frac{q_1' - \mu}{\gamma_1(N_1 - q_1)} + h_1 - \lambda_1$ (A.18)

Since it can be reasonably expected that $p_2 > p_1$, we can set μ=0. Then, (A.18) becomes:

$p_1 = \frac{q_1'}{\gamma_1(N_1 - q_1)} + h_1 - \lambda_1$ (A.19)

Similarly, we have,

$p_2 = \frac{q_2'}{\gamma_2(N_2 - q_2)} + h_2 - \lambda_2$ (A.20)

Using (v) and (vi), we have:

$\lambda_1' = -[-(p_1 - h_1 + \lambda_1)(\alpha_1 - \gamma_1 p_1) + \beta_2 * (N_2 - q_2) * (p_2 - h_2 + \lambda_2)]$

(A.21)

$\lambda_2' = -[-(p_2 - h_2 + \lambda_2)(\alpha_2 - \gamma_2 p_2 + \beta_2 q_1)]$

(A.22)

Differentiating (A.19), we get,

$p_1' = \frac{d}{dt}\left\{\frac{q_1'}{\gamma_1(N_1 - q_1)}\right\} - \lambda_1'$, or

$\gamma_1 p_1' = \dfrac{d}{dt}\left\{\dfrac{q_1'}{(N_1 - q_1)}\right\} - \gamma_1 \lambda_1'$

$\gamma_1 p_1' = \dfrac{d}{dt}\{(\alpha_1 - \gamma_1 p_1)\} - \gamma_1 \lambda_1'$

$\gamma_1 p_1' = -\gamma_1 p_1' - \gamma_1 \lambda_1'$

$$2p_1' = -\lambda_1' \tag{A.23}$$

From (A.21), and using expressions from (A.19) and (A. 20), we can write,

$$\lambda_1' = -[-(p_1 - h_1 + \lambda_1)(\alpha_1 - \gamma_1 p_1) + \beta_2 * (N_2 - q_2) * (p_2 - h_2 + \lambda_2)]$$

$$\lambda_1' = -\left[-\left\{\frac{q_1'(\alpha_1 - \gamma_1 p_1)}{\gamma_1(N_1 - q_1)}\right\} + \beta_2 * \frac{q_2'}{\gamma_2}\right]$$

$$\lambda_1' = -\left[-\left\{\frac{(q_1')^2}{\gamma_1(N_1 - q_1)}\right\} + \beta_2 * \frac{q_2'}{\gamma_2}\right] \tag{A.24}$$

Putting (A.24) in (A.23), we get,

$$2p_1' = \left[-\left\{\frac{(q_1')^2}{\gamma_1(N_1 - q_1)}\right\} + \beta_2 * \frac{q_2'}{\gamma_2}\right] \tag{A.25}$$

This gives the required for Equation (40). Equation (41) can be derived similarly.

Proposition 4:

Objective Function

Maximize

$$\pi = \{m_1 q_1' + m_2 q_2' - x_1 - x_2\}dt \tag{42}$$

Subject to Constraints

$$q_1' = (N_1 - q_1) * (\alpha_1 ln\, x_1) \tag{43}$$

$$q_2' = (N_2 - q_2) * (\alpha_2 ln\, x_2 + \beta_2 q_1) \tag{44}$$

$$x_1 + x_2 \leq X \tag{45}$$

$$0 \leq q_1 \leq N_1, \quad 0 \leq q_2 \leq N_2 \tag{46}$$

From Equation (48), we have a Lagrangean function as given below:

$$L = \{(m_1 + \lambda_1)q_1' + (m_2 + \lambda_2)q_2' - x_1 - x_2\} - \mu(x_1 + x_2 - X)$$

The following optimality conditions are used in this case:

$$(i)\frac{\partial L}{\partial x_1} = 0, \quad (ii)\frac{\partial L}{\partial x_2} = 0\,, \quad (iii)\,\mu \geq 0,$$

$$(iv)\mu(x_1 + x_2 - X) = 0, \quad (v)\lambda_1' = -L_{q_1}, \quad (vi)\ \lambda_2' = -L_{q_2}$$

Condition (i) gives the following equation

$\frac{(m_1+\lambda_1)(N_1-q_1)\alpha_1}{x_1} - 1 - \mu = 0$, or

$$(m_1 + \lambda_1)(N_1 - q_1)\alpha_1 - (1+\mu)x_1 = 0 \qquad \text{(A.26)}$$

$$(m_2 + \lambda_2)(N_2 - q_2)\alpha_2 - (1+\mu)x_2 = 0 \qquad \text{(A.27)}$$

Condition (v) gives:

$$\lambda_1' = -[(m_1 + \lambda_1)(-\alpha_1 ln\, x_1) + (m_2 + \lambda_2)\beta_2(N_2 - q_2)] \qquad \text{(A.28)}$$

From (A.26), we get,

$$x_1 = \frac{(m_1+\lambda_1)(N_1-q_1)\alpha_1}{(1+\mu)} \qquad \text{(A. 29)}$$

Differentiating (A.29) with respect to *t*, we get,

$$x_1' = -\frac{(m_1+\lambda_1)\alpha_1}{(1+\mu)} * q_1' + \frac{(N_1-q_1)\alpha_1}{(1+\mu)} * \lambda_1' \qquad \text{(A.30)}$$

Putting the values of q_1' and λ_1' from Equations (43) and (A.28), respectively, we get the required form in Equation (49). Equation (50) can be derived similarly.