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A Cardinality Induced Extended Formulation of the Single-Source Un-capacitated Facility Location Problem and its Polyhedra

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In this paper, an new extended formulation of the Single-Source Un-capacitated Facility Location Problem (SSUFLP) is presented that incorporates the cardinality of the customer set assigned to facilities (or agents) into its formulation. Given a set of M potential facilities and N customers (or jobs), the traditional integer programming formulation consists of O(mn) variables and constraints, where $|M| = m$ and $|N| = n$. In our extended formulation, potential facility location variables as well as variables describing assignment of customers to agents are disaggregated into n possible cardinalities. Consequently, our formulation consists of O(mn^2) variables and constraints. Given this, we first show that all *non-trivial* facets of the polytope associated with this disaggregated formulation can be described by 0-1 coefficients for variables representing assignment of customers to agents and non-negative, integer coefficients of variables representing facility location. We next present in detail, all possible structures of these non-trivial facet inequalities, which we refer to as p-Agent Cardinality Matching inequalities. These inequalities are constructed around $N' \subseteq N$ jobs assigned to $W_p \subseteq M$ agents in which $|W_p| = p$. This is motivated by identifying a fractional solution to the LP relaxation of the extended formulation, in which all the fractional variables are associated with N' and W_p. The basic idea behind these inequalities is to 'match' $n' = |N'|$ jobs to a set of $2 \leq p \leq m$ agents with specific cardinalities. The structure varies depending on the relative sizes of p and n', as well as the cardinalities associated each agent in W_p. All the structures of the p-Agent Cardinality Matching inequalities presented in this paper are shown to be facets of the polytope defined by the convex hull of feasible solutions to our extended formulation. For each such structure, we identify fractional solutions to the LP relaxation of our formulation that violate it. These structures cover all possible combinations of N' and W_p. Therefore, the p-Agent Cardinality Matching inequalities along with the trivial inequalities completely describe the polytope of the LP relaxation of the extended formulation.

Keywords: Integer Programming, Facility Location, Valid Inequalities, Facets.

1. Introduction

The Single-Source Un-capacitated Facility Location Problem (SSUFLP) (also referred to as the Simple Plant Location Problem) is a well-considered problem in Integer Programming. In SSUFLP, a set of M potential facility locations (or agents) and a set of N customers (or jobs) are specified. The cost of setting up a facility at location $i \in M$ is f_i and the cost of servicing a customer $j \in N$ entirely by a facility at $i \in M$ is c_{ij}. Without loss of generality, we can assume the cost parameters to be non-negative. The optimization problem is to determine which agents to open, and to which opened facility must each customer $j \in N$ must be assigned, so as to minimize total cost. The standard integer programming formulation of SSUFLP involves two sets of binary variables: $y_i = 1$ if facility at location $i \in M$ is opened, 0 otherwise, and $x_{ij} = 1$ if customer j is assigned to facility at location i, 0 otherwise. Such a formulation of SSUFLP is:

(**P**$_{xy}$)

$$Minimize\ F(x,y) = \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{i \in M} f_i y_i$$

Subject to:

$$\sum_{i \in M} x_{ij} = 1 \qquad \forall j \epsilon N \qquad (1)$$

$$x_{ij} \leq y_i \qquad \forall i \in M,\ j \in N \qquad (2)$$

$$x_{ij}, y_i \in \{0,1\} \qquad \forall i \in M, \quad j \in N. \qquad (3)$$

In (**P**$_{xy}$), (1) describes the assignment of each customer $j \in N$ to one of the locations in M and is known as *semi-assignment* constraints, while (2) (known as Variable Upper Bound (VUB) constraints) ensures that customer j cannot be assigned to a facility location i if it is not set up. SSUFLP is NP-Hard [10], and therefore there has been considerable interest in evolving methods to solve large instances of it in reasonable time. One such approach has been to try and describe the polyhedron defining the convex hull of feasible solutions to (**P**$_{xy}$) as 'closely' as possible. The main thrust of this paper is to start with a extended formulation of SSUFLP that in turn reveals inequalities that are facets of the associated polyhedron, and in all cases, along with trivial facets, completely describes it.

1.1 Literature Review

Spanning almost four decades, extensive work has been done in attempting to solve SSUFLP and its closely related problems such as the capacitated facility location problem and the capacitated concentrator location problem. The latter two includes knapsack type constraints on each agent i. It is important to underscore the fact that the capacitated concentrator location problem is the capacitated version of SSUFLP wherein every customer is served wholly by one concentrator. We will first review, albeit briefly, work done on the capacitated version of SSUFLP, followed by that on SSUFLP in greater detail.

There are two broad and yet distinct methodological approaches to solving the capacitated version of SSUFLP. One approach involved the use of Lagrangian relaxation, particularly with respect to the capacitated version of SSUFLP. Sridharan's [16] work on the capacitated version of SSUFLP is amongst the earliest known, followed more recently by Holmberg et al. [13], Cortinhal and Captivo [8], Chen and Ting [4] and several in between. The main thrust of this approach has been to dualize the semi-assignment constraints and solve a series of knapsack constraints to obtain a tight lower bound. In addition, primal heuristics, including Ant Colony approach in [4] have been used to obtain good upper bound. This is then embedded in a branch-and-bound to get the exact solution. The other approach involves attempting to describe the polytope defined by the convex hull of feasible solutions to the capacitated version of SSUFLP. Aardal [1] considered knapsack cover, flow cover and effective capacity inequalities that specifically address the presence of knapsack inequalities in capacitated facility location. In addition, a form of *combinatorial* inequalities was introduced for SSUFLP that is also valid for the capacitated version. Labbè and Yaman [14] introduced the quadratic form of the Capacitated Concentrator Location Problem, with a formulation that involves constraints which are quadratic. They studied the polytope of the resulting formulation and developed strong inequalities for it. These inequalities were incorporated as cuts in a branch-and-cut methodology using separation heuristics. Yang et a. [20] considered the $(\mathbf{P}_{xy})$ given above along with the knapsack constraints for each $i \in M$. In their approach, they introduced Lifted Cover Inequalities (LCI) and Fenchel cutting planes (FCI) that arise from the knapsack constraints. They implemented exact separation algorithms for both. Further, they implemented a cut-and-solve approach, with branching done on a sum of variables, akin to a GUB constraint. Gouveia and Saldanha-da-Gama [11] considered a variant of the Capacitated Concentrator Location Problem, wherein the knapsack constraints are replaced by GUB constraints that limit the number of customers assigned to concentrators. They further considered an extension wherein several capacity options can be chosen at each concentrator location. For this problem, they presented an extended formulation that disaggregates the y variables into various cardinalities, each representing the number of terminals assigned to it. They also presented "≤" and "≥" inequalities for their extended formulation.

Galli et. al [9] provide a comprehensive exposition on the prior reported work on describing the SSUFLP polyhedra defined as: $H(x, y) = Conv\{(x, y) \in R^{mn+m} |\ (1)\text{-}(3)\}$, the convex hull of feasible solutions to (1)-(3), which we provide here more briefly. Cornuéjols and Thizy [7] represented $(\mathbf{P}_{xy})$ as a *vertex packing* problem. Using known results for the vertex packing problem, they established the dimension of $H(x, y)$ and that VUB constraints (2), $x_{ij} \geq 0$ and $y_i \leq 1$, are trivial facets of $H(x, y)$. Apart from the results presented in Galli et. al [9], what is indeed common across all the work presented below is that the vertex packing problem is represented by a graph $G(V, E)$. Further, any new valid inequality is defined on $G(V, E)$ with a specific structure. Using the terminology in [9], Cornuéjols and Thizy [7] presented *circulant* inequalities as generalizations of those in [12]. As well, they presented translates of the *odd hole* inequalities for the vertex packing problem as valid inequalities for $H(x, y)$. It has been shown that given the structure of constraints describing the vertex packing problem and therefore ($\mathbf{P}_{xy}$), that any non-trivial facet of $H(x, y)$ is of the form $\alpha^{T}x \leq \beta^{T}y+\gamma$, with $\alpha\geq 0$, $\beta\geq 0$ and $\gamma>0$. Cho et. al [5, 6], to begin with, examined inequalities with elements in α and β being binary. Aardal [1] referred to them as *combinatorial* inequalities. Combinatorial inequalities are generalizations of circulant inequalities, which in turn generalize odd cycle inequalities. Cho et. al [5] showed that a combinatorial inequality is a facet of $H(x, y)$ if and only if a) it constitutes a non-empty face of $H(x, y)$, b) it has at least three non-zero elements in β, and c) no element in α, can be lifted from 0 to 1. In addition, all non-trivial facets of $H(x, y)$ needed to completely describe its polytope for the cases, i) $m = 3$, $n \geq 3$ and ii) $m \geq 3$, $n = 3$, were identified in [5] and [6], respectively. All of them are combinatorial inequalities. Finally, Cho et. al [6] presented a class of inequalities in which components of α are either 0, 1 or 2, components of β is binary and $\gamma = 2$. Aardal [1] presented a special class of combinatorial inequalities which are a generalization of circulant inequalities.

Cánovas et al. [2] introduced *grille* inequalities in which the elements of α are binary, $\gamma = 1$, but elements of β can be arbitrary non-negative integers. Like circulant inequalities, grille inequalities are derived from the associated vertex packing graph having a specific structure. They have been shown to be facets of $H(x, y)$. In [3], the same authors introduced *fan* and *wheel* inequalities which are not combinatorial. As with other inequalities mentioned earlier, these inequalities are derived from the vertex packing graph, with the graph resembling a fan and wheel, respectively. Galli et. al [9] introduced a new way of constructing valid inequalities for $H(x, y)$, the resulting inequalities they refer to as *homogenous* inequalities. Unlike circulant, grille, fan or wheel inequalities, homogenous inequalities do not specify a structure on the associated vertex packing graph $G(V, E)$. Rather, given any connected graph, with certain characteristics there is a procedure to construct strong homogenous inequalities. In that respect, homogenous inequalities are shown to be generalizations of combinatorial inequalities as well as grille inequalities. In particular, they point to the existence of facet defining homogenous inequalities which are neither circulant nor grille inequalities. The number of such inequalities are exponential in number. Finally, they present a new procedure called *facility augmentation*, using which many more facet defining inequalities of $H(x, y)$ can be derived.

1.2 Contributions of this Paper

The focus of this paper is on a new extended formulation of SSUFLP and it associated valid inequalities. The principal idea behind this formulation is to specify a cardinality to a facility that is opened, where cardinality defines the number of customers assigned to it. Similarly, the assignment of a customer to a facility is also disaggregated by the associated cardinality of the facility to which it is assigned to. Accordingly, each y variable in ($\mathbf{P}_{xy}$) is disaggregated into n variables, each associated with a cardinality varying from 1 to n. Similarly, each x variable in ($\mathbf{P}_{xy}$) is disaggregated into n distinct variables. Consequently, the number of variables and constraints expand to $O(mn^2)$. In Section 2.0, we first formally present the extended formulation, denoted ($\mathbf{P}_{zy}$). We next identify all the trivial facets of the polytope defined by the convex hull of the feasible solutions to ($\mathbf{P}_{zy}$). Finally, we discuss in detail the essential features of any *non-trivial* facet of the convex hull of the feasible solutions to (**P_{zy}**). Most importantly, we show that all non-trivial facets of the polytope associated with this disaggregated formulation can be described entirely in terms of 0-1 coefficients for variables representing assignment of customers to agents and non-negative, integer coefficients of variables representing facility location. This greatly simplifies the identification of all the non-trivial facets. We refer to these non-trivial facet inqualities as *p*-Agent Cardinality Matching (*p*-ACM) inequalities, which is described in depth in Section 3. The *p*-ACM inequalities are defined by $N' \subseteq N$ jobs assigned to $W_p \subseteq M$ agents in which $|W_p| = p$. This in turn is motivated by isolating the fractional part of any feasible solution to the LP relaxation of our extended formulation. That is, all the variables which are non-integer, are associated with N' and W_p, which are rendered infeasible by the *p*-ACM inequality. The structure varies depending on the relative sizes of p and n', as well as the cardinalities associated each agent in W_p. All the structures of the *p*-Agent Cardinality Matching inequalities presented in Section 3.0 are shown to be facets of the polytope defined by the convex hull of feasible solutions to our extended formulation. For each such structure, we identify fractional solutions to the LP relaxation of our formulation that violate it. Thus, these structures cover all possible combinations of N' and W_p. Therefore, the *p*-Agent Cardinality Matching inequalities along with the trivial inequalities completely describe the polytope of the LP relaxation of the extended formulation. This then is the principal contribution of this paper.

2.0 An Extended Formulation of SSUFLP and its Polytope

The formulation proposed below disaggregates each y variable in (**P_{xy}**) into n variables, each specifying the number of customers (jobs) assigned to the agent that the y variable represents. In the same way, the x variables in (**P_{xy}**) are also disaggregated in terms of the cardinality associated with agent i. The binary variables used are: $y_{ik_i} = 1$, if facility (agent) i is opened with k_i jobs assigned to it with $1 \le k_i \le n$, 0 otherwise, and $z_{ijk_i} = 1$, if job j is one of the k_i jobs assigned to agent i, 0 otherwise. The extended formulation is:

(P_{zy})

$$Minimize \qquad F(z,y) = \sum_{i\in M}\sum_{j\in N}\sum_{k_i=1}^{n} c_{ij} z_{ijk_i} + \sum_{i\in M}\sum_{k_i=1}^{n} f_i y_{ik_i}$$

s.t.

$$\sum_{i\in M}\sum_{k_i=1}^{n} z_{ijk_i} = 1 \qquad \forall j \in N \qquad (4)$$

$$z_{ijk_i} \le y_{ik_i} \qquad \forall i \in M, j \in N, k_i = 1, \dots \dots, n-1 \qquad (5)$$

$$z_{ijn} = y_{in} \qquad \forall i \in M, j \in N \qquad (6)$$

$$\sum_{j \in N} z_{ijk_i} = k_i y_{ik_i} \qquad \forall i \in M, k_i = 1, \dots \dots, n-1 \qquad (7)$$

$$\sum_{k_i=1}^{n-1} y_{ik_i} + \sum_{i'=1}^{m} y_{i'n} \leq 1 \qquad \forall i \in M \qquad (8)$$

$$z_{ijk_i}, y_{ik_i} \in \{0,1\} \qquad \forall i \in M, j \in N, k_i = 1, \dots \dots, n \qquad (9)$$

In the formulation above, (4) represents the *semi-assignment* constraints for each $j \in N$ across all agents and cardinalities, (5) the VUB constraints associated with each $i \in M$, $j \in N$ and $1 \leq k_i \leq n$-1. Note that if agent i is opened with all n jobs assigned to it, then the VUB constraints become equalities as indicated in (6). Constraints (7) enforce the *cardinality* requirement that if a agent i with a cardinality of k_i is opened, then exactly k_i jobs have to be assigned to it. Constraints (8) model two phenomena. The first is that for each agent $i \in M$, at most one type of cardinality is opened. In addition, if some other agent $i' \notin i$ with a cardinality of n is opened, i.e., all jobs are assigned to i', then agent i of any cardinality cannot be open.

Note that (**P**$_{xy}$) and (**P**$_{zy}$) are not equivalent. While every feasible solution in (**P**$_{zy}$) can be translated to an equivalent feasible solution in (**P**$_{xy}$) with the same objective function value, the converse is not true. Specifically, (**P**$_{xy}$) allows for a agent i to be opened ($y_i = 1$), without any job being assigned to it. Such an option does not exist in (**P**$_{zy}$), which requires at least one job to be assigned for a agent to be opened. However, it is worth noting, that as long as $f_i \geq 0$ (which is generally the case), the optimal solution to (**P**$_{xy}$) will always be one wherein if a agent is opened, at least one job will be assigned to it. This fact will have some bearing when we examine the LP relaxations of (**P**$_{xy}$) and (**P**$_{zy}$), respectively.

Let,

$LP(x, y) = \{(x, y) \in R^{mn+m} \,|(1)\text{-}(2), x \geq 0, 0 \leq y \leq 1, \text{ are satisfied}\}$, while (10)

$LP(z, y) = \left\{(z, y) \in R^{mn^2+mn} | (4) - (8), z \geq 0, y \geq 0\right\}$. (11)

Consider a $(z^+, y^+) \in LP(z, y)$. A solution (x^+, y^+) can be constructed from (z^+, y^+) as follows:

$$x_{ij}^+ = \sum_{k_i=1}^{n} z_{ijk_i}^+ \quad and \quad y_i^+ = \sum_{k_i=1}^{n} y_{ik_i}^+, \qquad for\ each\ i \in M, j \in N. \qquad (12)$$

It is easy to see that $(x^+, y^+) \in LP(x, y)$. This is because aggregation of z over k_i ensures that (4) reduces to (1), while the aggregation of z and y as shown in (12) ensures that (5) and (6), collapse to (2). Finally, (8) and (4) together ensure that $0 \leq x_{ij} \leq 1$ and $0 \leq y_i \leq 1$ for each $i \in M$ and $j \in N$. Further, $F(z^+, y^+) = F(x^+, y^+)$. Does every $(x^+, y^+) \in LP(x, y)$ translate to a $(z^+, y^+) \in LP(z, y)$ with $F(z^+, y^+) = F(x^+, y^+)$? To answer this question, consider the solution type $(x^+, y^+) \in LP(x, y)$ in which $y_i^+ = Max\{x_{ij}^+ | j \in N\}$ for each $i \in M$. Let $J_i^+ = \{j \in N | x_{ij}^+ > 0\}$. An equivalent $(z^+, y^+) \in LP(z, y)$ can be constructed in the following way. Let $\Delta = Min\ \{x_{ij}^+ |\ j \in J_i^+\}$ and $k = |J_i^+|$. Set i) $z_{ijk}^+ = \Delta$ for each $j \in J_i^+$, ii) $y_{ik}^+ = \Delta$, iii) $y_i^+ = x_{ij}^+ - \Delta$ for each $j \in J_i^+$, and finally iv) redefine $J_i^+ = \{j \in N | x_{ij}^+ > 0\}$. Repeat steps i) – iv) till $J_i^+ = \phi$. An important observation about this process is that (z^+, y^+) so constructed satisfy the cardinality constraints (7) without violating (5). Further, $F(z^+, y^+) = F(x^+, y^+)$. However, if $1 \geq y_i^+ > Max\{x_{ij}^+ | j \in N\}$ for one or more $i \in M$, it is no longer possible to find an equivalent $(z^+, y^+) \in LP(z, y)$ such that $F(z^+, y^+) = F(x^+, y^+)$.

Example 1: Consider a $(x^+, y^+) \in LP(x, y)$ in part where $y_i^+ = 0.7$, $x_{i1}^+ = x_{i2}^+ = x_{i3}^+ = 0.7$, $x_{i4}^+ = x_{i5}^+ = 0.5$, $x_{i6}^+ = 0.3$. The corresponding (z^+, y^+) obtained would be $y_{i6}^+ = z_{i1,6}^+ = z_{i2,6}^+ = z_{i3,6}^+ = z_{i4,6}^+ = z_{i5,6}^+ = z_{i6,6}^+ = 0.3$, $y_{i5}^+ = z_{i1,5}^+ = z_{i2,5}^+ = z_{i3,5}^+ = z_{i4,5}^+ = z_{i5,5}^+ = 0.2$, $y_{i3}^+ = z_{i1,3}^+ = z_{i2,3}^+ = z_{i3,3}^+ = 0.2$. Observe that constraints (5), (7) and (8) are all satisfied. However, if $y_i^+ = 1.0$, no equivalent solution in $LP(z, y)$ exists.

Since, typically $f_i \geq 0$ for all $i \in M$, one can expect the optimal solution to the LP relaxation of ($\mathbf{P}_{xy}$) to satisfy $y_i = Max\ \{x_{ij} | j \in N\}$ for each $i \in M$. Therefore, the initial LP relaxations of ($\mathbf{P}_{zy}$) and ($\mathbf{P}_{zy}$) will give the same lower bound value. However, the significance of the above observation comes to play upon adding cuts to either LP relaxations. Adding cuts to $LP(x, y)$ can result in fractional solutions wherein $y_i^+ > Max\{x_{ij}^+ | j \in N\}$ for one or more $i \in M$. Since no equivalent solution exists in $LP(z, y)$, the bounds obtained from the LP relaxation of ($\mathbf{P}_{zy}$) upon adding the 'same' cuts become superior.

Example 2: Consider the following partial fractional solution in $LP(x, y)$: $y_1 = y_2 = y_3 = 0.5$, $x_{11} = x_{13} = x_{21} = x_{22} = x_{32} = x_{33} = 0.5$, which in $LP(z, y)$ translates to $y_{12} = y_{22} = y_{32} = 0.5$, $z_{112} = z_{132} = z_{212} = z_{222} = z_{322} = z_{332} = 0.5$. However, if the classic odd-hole inequality $x_{11} + x_{13} + x_{21} + x_{22} + x_{32} + x_{33} \leq y_1 + y_2 + y_3 + 1$ is added to $LP(x, y)$, then the fractional solution can adjust to $y_1 = y_2 = y_3 = 2/3$, $x_{11} = x_{13} = x_{21} = x_{22} = x_{32} = x_{33} = 0.5$, thereby satisfying the odd-hole inequality and also feasible in $LP(x, y)$. However, the corresponding fractional solution, $y_{12} = y_{22} = y_{32} = 2/3$, $z_{112} = z_{132} = z_{212} = z_{222} = z_{322} = z_{332} = 0.5$, does not belong in $LP(z, y)$, as the cardinality constraints (7) are violated. Herein lies the value of the disaggregated formulation.

2.1 Trivial Facets of the ($\mathbf{P}_{zy}$) *Polyhedra*

Let

$$H(z, y) = Conv\{(z, y) \in B^p | (z, y)\ satisfies\ (4) - (9), p = mn^2 + mn\} \quad (13)$$

In order to keep our exposition as general as possible, we will assume that $m \geq 3$. Clearly, for $m = 1$, ($\mathbf{P}_{zy}$) is trivial. In the case of $m = 2$, one can construct an equivalent problem by adding another agent i' with $f_{i'} = \infty$.

In order to facilitate our subsequent discussions on the dimension of $H(z, y)$ and its facets, we present the notion of a *cyclic* (k, l) *matrix* as described in [3]. It is a square matrix of dimension kXk, with entries of 0 or 1. For rows $i < k$, starting from column i, l consecutive columns consist of ones, and the rest zero. The l consecutive positions include the cycling back to the first (i+l-1-k) columns consisting of ones when (i+l-1-k)>0. For row $i = k$, if either k or l is odd, then the same rule applies as far as positioning of ones are concerned. However, if both k and l are even, then row k is modified wherein the element in the first column consists of a zero followed by l-1 consecutive ones. With such a structure, the rows in the *cyclic* (k, l) *matrix* are linearly independent.

Lemma 2.1 $Dim\{H(z, y)\} = mn(n-1) + m-1$

Proof: It suffices to identify $mn(n-1)+m$ affinely independent solutions $(z, y)\in H(z, y)$ noting that $(\mathbf{0}, \mathbf{0})\notin H(z, y)$. Assume items in M to be indexed arbitrarily from $i = 1, \ldots, m$. For each $1\leq i\leq m$ and for each $1\leq k_i\leq n-1$, the solution set is: $y_{ik_i} = 1$, $z_{ijk_i} = 1$ for $j\in N_{k_i}$, $y_{i+1,k_{i+1}} = 1$, $z_{i+1,j,k_{i+1}} = 1$ for $j\in\{N\text{-}N_{k_i}\}$, where $N_{k_i}\subset N$ is a selection with $|N_{k_i}| = k_i$, $k_{i+1} = n\text{-}k_i$, and the rest of the variables equal to zero, while noting that when $i = m$, we replace index i+1 with 1. Observe that the above 'block' of solutions follows the *cyclic* (m, 2) *matrix*. For a given i and $k_i \leq n-1$, a set of n linearly independent selections of N_{k_i} from N can be made using the *cyclic* (n, k_i) *matrix*. Thus, by considering n selections of N_{k_i} and varying i, one gets mn solutions. By varying k_i from 1 to n-1, a total of $mn(n-1)$ solutions are obtained. Finally, for each $i\in M$, $y_{in} =1$, $z_{ijn} = 1$ for each $j\in N$, which are m in number, giving a total of $mn(n-1)+m$. □

Lemma 2.2 The inequalities, a) $z_{ijk_i} \geq 0$ for all $i\in M$, $j\in N$, $1\leq k_i\leq n$, and b) VUB constraints (5) are all *trivial* facets of $H(z, y)$.

Proof: For both inequalities, $mn(n-1)+m-1$ affinely independent solutions in $H(z, y)$ that satisfy the respective inequalities as an equality are identified.

Consider first the inequality in a). When $k_i = n$, all the solutions listed in the proof of Lemma 2.1 satisfy $z_{ijn} \geq 0$ as an equality, except for the solution, $y_{in} =1$, $z_{ijn} = 1$ for each $j\in N$, resulting in $mn(n-1)+m-1$ affinely independent solutions in $H(z, y)$.

Next consider cases in which $k_i \leq n-1$. The set of affinely independent solutions are described pictorially in Figure 1 in which columns represent z variables and rows represent feasible solutions that satisfy a) as an equality. The columns under label $i(j, k_i)$ represent the collection of z variables in which i is fixed, but j varies from 1 to n and k_i varies from 1 to n-1. Each 'block' in the figure represents the intersection of those rows and columns in which at least one z variable is non-zero. The z variables are ordered such that the first set of columns come under $i(j, k_i)$, followed by columns associated with indices in $\{M\text{-}i\}$ arranged in any arbitrary order as: $i_1, \ldots, i_{m-1}$.

The set of solutions that satisfy $z_{ijk_i} \geq 0$ as an equality are as follows:

I) One set involves pairings $i_l(j, k_{i_l})$ and $i_{l+1}(j, k_{i_{l+1}})$, for $l = 1,\ldots, m-2$. In addition, when $l = m-1$, then l+1 is equal to 1 if m is even, and 2, otherwise. In Figure 1, the block rows labeled as A1(i_1 and i_2), A1(i_2 and i_3) and A1(i_1 and i_3), display such an arrangement. It is clear from the figure, that these rows display a (m-1, 2)-*cyclic matrix* structure. Consequently, for these rows to be affinely independent, it suffices to show that the rows in each block are independent. The solution in each row is: $y_{i_lk_{i_l}} = 1$, $z_{i_ljk_{i_l}} = 1$ for $j\in N_{k_{i_l}}$, $y_{i_{l+1}k_{i_{l+1}}} = 1$, $z_{i_{l+1}jk_{i_{l+1}}} = 1$ for $j\in N_{k_{i_{l+1}}}$, where $N_{k_{i_l}}\subset N$, $N_{k_{i_{l+1}}} = \{N - N_{k_{i_l}}\}$, with $|N_{k_{i_l}}| = k_{i_l}$ and $|N_{k_{i_{l+1}}}| = n\text{-}k_{i_l}$. The cardinality k_{i_l} takes values from 1 to n-1. For each value of k_{i_l}, n independent selections of $N_{k_{i_l}}$ from N are made using the (n, k_{i_l})-*cyclic matrix*, hence affinely independent. These set of solutions account for a total of (m-1)n(n-1) solutions.

II) Another set of solutions are: i) $y_{ik'_i} = 1$, $z_{ijk'_i} = 1$ for $j \in N_{k'_i}$, $y_{i_1 k_{i_1}} = 1$, $z_{i_1 j k_{i_1}} = 1$ for $j \in N_{k_{i_1}}$, with $N_{k'_i} \subset N$, $N_{k_{i_1}} = \{N - N_{k'_i}\}$ and $|N_{k'_i}| = k'_i$. Here, the cardinality k'_i takes on all values between 1 and n-1, except k_i. For each value of k'_i, n independent selections of $N_{k'_i}$ from N are made. ii) $y_{ik_i} = 1$, $z_{ij'k_i} = 1$ for $j' \in N_{k_i}$, $y_{i_1 k_{i_1}} = 1$, $z_{i_1 j' k_{i_1}} = 1$ for $j' \in N_{k_{i_1}}$, where $N_{k_i} \subset \{N\text{-}j\}$, $N_{k_{i_1}} = \{N - N_{k_i}\}$. Here, n-1 independent selections of N_{k_i} from $\{N\text{-}j\}$ are made. These set of solutions are represented in rows labeled, A1(i and i_1) in Figure 1, and amount to n(n-2)+(n-1) in number.

III) Finally, the solution set, $y_{in} = 1$, $z_{ijn} = 1$ for all $j \in N$ and the rest of the variables equal to zero also satisfies a) as an equality. There are m such solutions, one for each $i \in M$. Hence, a total of mn(n-1)+m-1 independent solutions have been identified.

Consider next the VUB constraints (5) in b). Observe that solution sets listed in I), II) i) and III) above for the inequality in a) all correspond to $y_{ik_i} = z_{ijk_i} = 0$ and satisfy (5) as an equality. These three sets together account for (m-1)n(n-1)+n(n-2)+m solutions. The solution set in II) ii) above is modified as follows: $y_{ik_i} = 1$, $z_{ijk_i} = 1$, $z_{ij'k_i} = 1$ for each $j' \in N_{k_i - 1} \subset \{N\text{-}j\}$, $y_{i_1 k_{i_1}} = 1$, $z_{i_1 j' k_{i_1}} = 1$ for $j' \in N_{k_{i_1}}$, where $|N_{k_i - 1}| = k_i$-1, $N_{k_{i_1}} = \{N\text{-}N_{k_i - 1}\}$. There are n-1 independent selections of $N_{k_i - 1}$ from $\{N\text{-}i\}$. This results in a total of mn(n-1)+m-1 affinely independent solutions. □

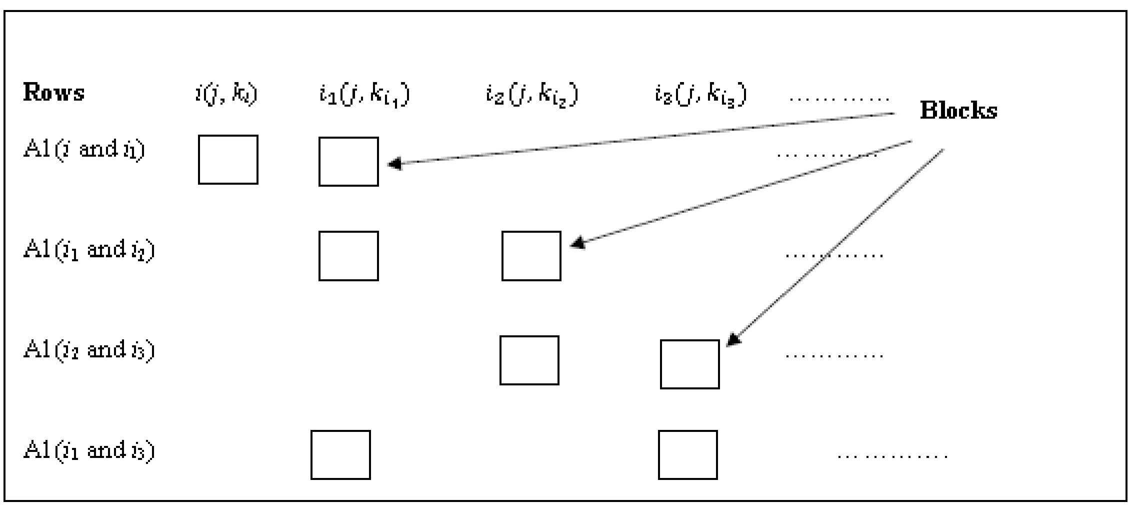

Figure 1. Illustration of a set of mn(n-1)-1 affinely independent solutions

Lemma 2.3 The inequality (8) for each $i \in M$ is a *trivial* facet of H(z, y).

Proof: To begin with, the solution set, $y_{i'n} = 1$, $z_{i'jn} = 1$ for each $j \in N$ satisfy (8) as an equality for each $i' \in M$, including $i' = i$. This accounts for m solutions. The rest of the feasible solutions that satisfy (8) as an equality are shown in Figure 2 below. One set of solutions, labeled as A2, comprise of two blocks, one associated with i and the other with each $i' \in \{M\text{-}i\}$. Specifically, $y_{ik_i} = 1$, $z_{ijk_i} = 1$ for $j \in N_{k_i}$, and $y_{i'k_{i'}} = 1$, $z_{i'jk_{i'}} =$

1 for $i' \in \{M\text{-}i\}$ and each $j \in \{N\text{-}N_{k_i}\}$. Here, $N_{k_i} \subset N$ is an independent selection with $|N_{k_i}| = k_i < n$ and $k_{i'} = |N\text{-}N_{k_i}|$. For each value of k_i, there are n independent selections of N_{k_i}. The n selections are structured using the *cyclic* (n, k_i) *matrix*. Further, k_i varies from 1 to n-1. Thus, A2 accounts for (m-1)n(n-1) solutions.

The third set of solutions, called A3, comprise of blocks associated with a pair $\{i_{m-1}, i_m\} \in \{M\text{-}i\}$ as well as i, shown in Figure 2. In A3, one set of solutions denoted A3(i), consists of a) $y_{ik_i} = 1,\ z_{ijk_i} = 1$ for each $j \in N_{k_i}$ with $N_{k_i} \subset N$, $|N_{k_i}| = k_i$ and $1 \le k_i \le n\text{-}2$, b) $y_{i_{m-1}1} = 1,\ z_{i_{m-1}j_{m-1}1} = 1$, for $j_{m-1} \in \{N\text{-}N_{k_i}\}$ and c) $y_{i_m k_{i_m}} = 1, z_{i_m j k_{i_m}} = 1$ for each $j \in \{N\text{-}N_{k_i}\text{-}j_{m-1}\}$. Here, n independent selections of N_{k_i} are made in a *cyclic* (n, k_i) *matrix* manner, with the columns of variables in N_{k_i} being contiguous. Thus, the i^{th} selection starts in column i and ends in column $i+k_i$-1. If ($i+k_i$-1) > n, then the last ($i+k_i$-1-n) variables chosen cycle back to the first ($i+k_i$-1-n) columns. Also, j_{m-1} is situated to the immediate right of the last column in N_{k_i}. Since k_i varies from 1 to n-2, there are n(n-2) solutions in A3(i). A second set of solutions, denoted A3(ii), consists of: a) $y_{i1} = 1,\ z_{ij1} = 1,\ y_{i_{m-1}k_{i_{m-1}}} = 1,\ z_{i_{m-1}j_{m-1}k_{i_{m-1}}} = 1$ for each $j_{m-1} \in N_{k_{i_{m-1}}}$ with $k_{i_{m-1}} = n - 2$ and b) $y_{i_m 1} = 1,\ z_{i_m j_m 1} = 1$ where $j_m \in \{N\text{-}j\text{-}N_{k_{i_{m-1}}}\}$. The n-2 columns under $N_{k_{i_{m-1}}}$ occupy contiguous positions to the immediate right of j, followed by j_m. Here, j is varied from 1 to n-1. Accordingly, the indices in $N_{k_{i_{m-1}}}$ and j_m shift to the right in a cyclic manner. Thus A3(ii) accounts for n-1 solutions satisfying (8) as an equality. The total number of solutions listed above equal to mn(n-1)+m-1.

By definition, the solutions listed above are shown to be affinely independent, by showing that the only way to combine these solutions to obtain a zero vector is by multiplying each by zero. The first observation is that the m solutions, each with a different agent having a cardinality of n are unique and therefore have to be multiplied by zero. Next observe that in Figure 2, the blocks under $i_2(j, k_{i_2})$ until $i_{m-2}(j, k_{i_{m-2}})$ (not in figure) are unique, displaying a staircase structure. Further, the solutions within each block under $i_2(j, k_{i_2})$ until $i_{m-2}(j, k_{i_{m-2}})$ are linearly independent as previously discussed. Hence, all solutions from A2(i and i_2) till A2(i and i_{m-2}) have to be multiplied by zero to obtain a zero vector. This leaves us with solutions in A2(i and i_{m-1}), A2(i and i_m) and A3(i, i_{m-1} and i_m). Again in A2(i and i_{m-1}), the part solutions under $i_{m-1}(j, k_{i_{m-1}})$ with $k_{i_{m-1}} = n - 1$ are unique. The same holds true for the part solutions in which $k_{i_{m-1}}$ varies from 2 to n-3. In A2(i and i_m), the part solutions under $i_m(j, k_{i_m})$ with $k_{i_m} = n - 1$ are unique as well. Thus, all these solutions have to be multiplied by zero to obtain a zero vector.

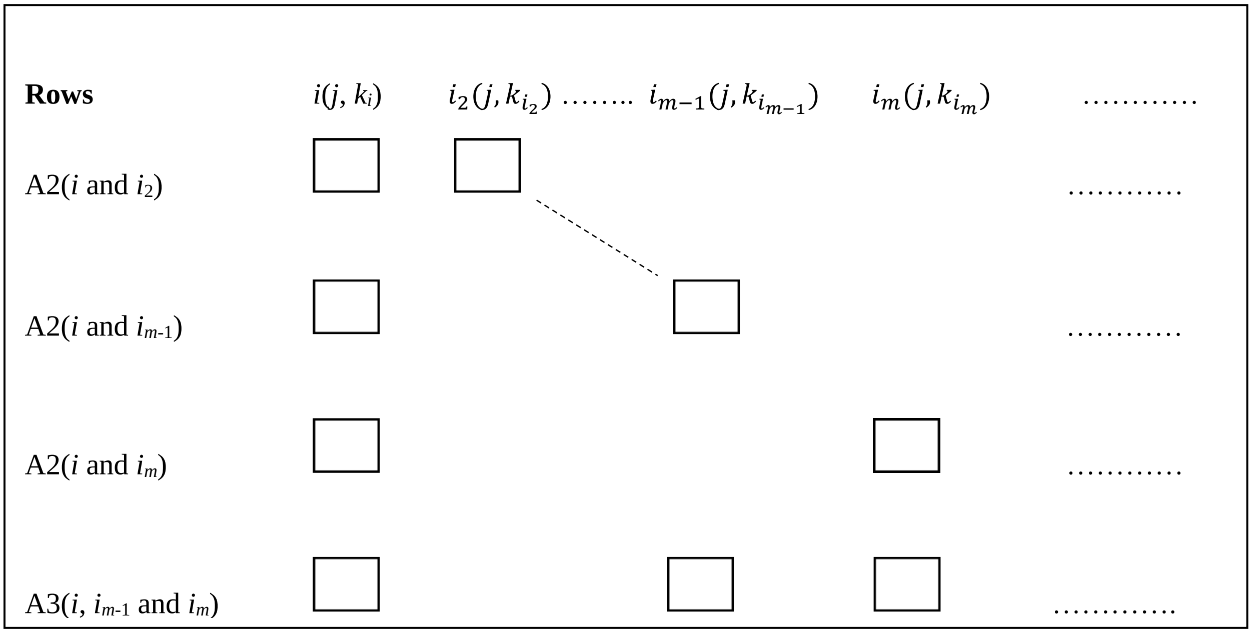

Figure 2. Illustration of independent solutions satisfying (8) as an equality

The solutions (or rows) left to be considered are shown in Figure 3 below. For the sake of brevity, only the cardinalities of part solutions are indicated. To obtain a zero vector from the remaining rows, we begin with the part solutions under $i_m(j, k_{i_m})$. To obtain a zero vector under $i_m(j, k_{i_m})$, the rows in A2(i and i_m) and A3(i, i_{m-1} and i_m) are combined as follows. First, each solution in A2(i and i_m) is weighted by 1. Second, each solution in A3(i) for $k_{i_m} = 2, \ldots, n-2$ is weighted by -1. Finally, for $k_{i_m} = 1$, the solutions in A3(i), consisting of n-1 rows that match those in A3(ii) are given a weight of -0.5, as are the rows in A3(ii). The remaining row in A3(i) with $k_{i_m} = 1$ is given a weight of -1. On applying the above weights, the columns under $i_m(j, k_{i_m})$ disappear, and A2(i and i_m) and A3(i, i_{m-1} and i_m) collapses into one row consisting of the following: i) $z_{ij1} = -1.5$ for each j corresponding to each row in A3(ii) and $z_{ij1} = -1$ for that j not in A3(ii), ii) $z_{ijn-1} = 1$ for each $j \in N_{n-1}$, for n selections of $N_{n-1} \subset N$, iii) $z_{ijn-2} = -0.5$ for the n-1 selections of $N_{n-2} \subset N$, iv) $z_{i_{m-1}j_{m-1}1} = -(n-2.5)$ for n-1 indices of j_{m-1}, and v) $z_{i_{m-1}j_{m-1}n-2} = -0.5$ for the same n-1 indices of j in N. Clearly, the solution set in A2(i and i_{m-1}) in which $z_{ij2} = 1$, for the n selections of $N_2 \subset N$ are unique under the columns $i(j, k_i)$ and therefore they are weighted by 0. Similarly under $i_{m-1}(j, k_{i_{m-1}})$, the collapsed row consisting of $z_{i_{m-1}j_{m-1}n-2} = -0.5$ for n-1 indices in N are unique and need to be weighted by 0. This then leaves us with n rows in A2(i and i_{m-1}) consisting of solutions $z_{ijn-1} = 1$ for each $N_{n-1} \subset N$, each of which is unique and needs to be weighted by 0. □

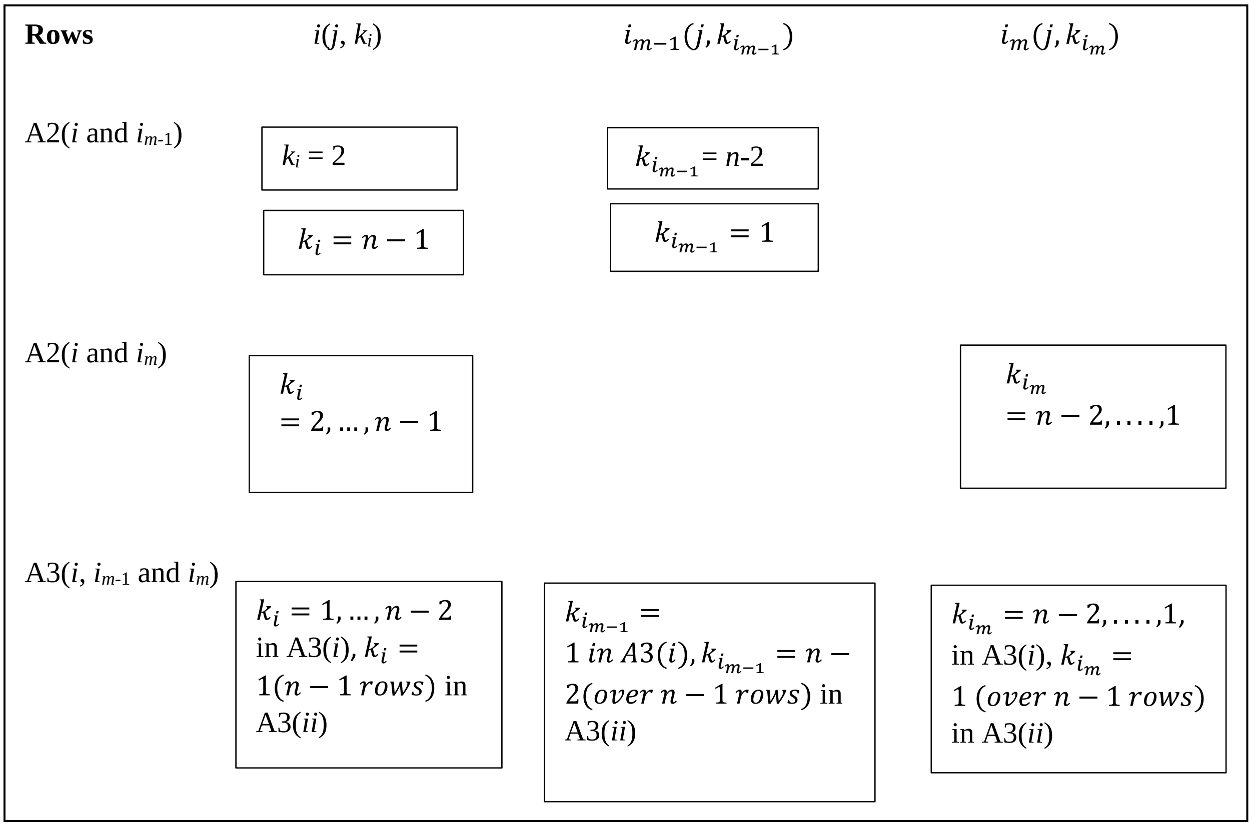

Figure 3. Illustration of the remaining rows in A2(i and i_{m-1}), A2(i and i_m) and A3(i, i_{m-1} and i_m).

2.2 Characterizations of non-trivial facets of H(z, y)

We next describe general properties of the facets of $H(z, y)$. Let α denote a vector consisting of sub-vectors, $(\alpha^1, \ldots, \alpha^m)$, with each α^i in turn consisting of sub-vectors $(\alpha^{i\text{-}1}, \ldots, \alpha^{i\text{-}n})$. Sub-vector $\alpha^{i-k_i} = (\alpha_{i1k_i}, \alpha_{i2k_i}, \ldots, \alpha_{ink_i})^T$ and $\alpha(j)$ is a vector consisting of coefficients α_{ijk_i}, with i varying from 1 to m, and k_i varying from 1 to n. β denotes a vector consisting of sub-vectors, $(\beta^1, \ldots, \beta^m)$, with $\beta^i = (\beta_{i1}, \beta_{i2}, \ldots, \beta_{in})$. In its most generic form, all facets of $H(z, y)$ take the form,

$$\alpha \mathbf{z} \leq \beta \mathbf{y} + \beta_0, \tag{14}$$

with β_0 being a scalar. Associated with any facet of $H(z, y)$, $S_{mn} = \{(z, y) \in B^{mn^2} | (z, y) \in H(z, y)$, that satisfy (14) as an equality and are affinely independent$\}$. From our discussion in Section 2.1, S_{mn} consists of $mn(n$-$1)+m$-1 affinely integer solutions. For notational convenience, $I_{mn} = \{1, \ldots, mn(n\text{-}1)+m\text{-}1\}$ denotes indices for each $(z, y) \in S_{mn}$. For each $l \in I_{mn}$, let $M'(l) \subseteq M$ denote the set of opened agents with $N_{k_i}(l) \subseteq N$, the set of k_i items assigned to $i \in M'(l)$. Clearly, $\sum_{i \in M'(l)} k_i = n$ and $\bigcup_{i \in M'(l)} N_{k_i}(l) = N$. Further, S_{mn} necessarily contains the m solutions: $y_{in} = 1$, $z_{ijn} = 1$ for all $j \in N$ for each $i \in M$. The remaining set in S_{mn} comprises of solutions in which $|M'(l)| \geq 2$.

Remark 2.1 An important observation about (14) is that the cardinality constraint (7) being an equality, (14) can be transformed into an equivalent inequality by changing some of the coefficients as follows:

i) $\alpha^{i-k_i} \rightarrow \alpha^{i-k_i} + \Delta,$ (15a)

ii) $\beta_{ik_i} \rightarrow \beta_{ik_i} + k_i\Delta,$ (15b)

for any $\Delta \neq 0$. Similarly, with the semi-assignment constraint (4), (14) can be transformed into an equivalent inequality by making the following changes,

i) $\alpha(j) \rightarrow \alpha(j) + \Delta,$ (16a)

ii) $\beta_0 \rightarrow \beta_0 + \Delta,$ (16b)

for any $\Delta \neq 0$.

Using the aforementioned transformations, any facet of $H(z, y)$ can be represented in a form wherein $(\alpha, \beta) \geq 0$ and $\beta_0 > 0$. Specifically, we will use those solutions corresponding to $l \in I_{mn}$ with $|M'(l)| \geq 2$ for the transformations. The following three-step procedure transforms a facet inequality of $H(z, y)$ into one in which $(\alpha, \beta) \geq 0$ and $\beta_0 > 0$.

***Algorithm_Transform*:**

- I) For any $i \in M$, and $k_i = 1,.., n$, whose $\beta_{ik_i} < 0$, execute (15a) and (15b) with $\Delta = -\beta_{ik_i}/k_i$.
- II) For any $i \in M$, $j \in N$ and $k_i = 1,..,n$, whose $\alpha_{ijk_i} < 0$, execute (16a) and (16b) with $\Delta = -\alpha_{ijk_i}$.
- III) For each $i \in M$, and $k_i = 1,..,n$-1, first determine $\alpha_{min} = Min\,\{\alpha_{ijk_i} | j \in N, \beta_{ik_i}/k_i\}$ and then execute (15a) and (15b) with $\Delta = -\alpha_{min}$.

It is important to note that in each of the three steps of ***Algorithm_Transform***, the left-hand-side (l-h-s) and right-hand-side (r-h-s) values of (14) change by the same amount for each $(z, y) \in S_{mn}$. Hence, after the transformation, (14) is satisfied exactly by each $(z, y) \in S_{mn}$, while still remaining valid. It is also clear from our earlier discussion in Section 2.1 that the m solutions: $y_{in} = 1$, $z_{ijn} = 1$ for each $j \in N$ are also part of S_{mn}. Therefore, without loss of generality, one can set $\alpha_{ijn} = \alpha_0$ and $\beta_{in} = n\alpha_0 - \beta_0$, for each $i \in M$, $j \in N$. Henceforth, our primary focus will be on coefficient vectors α^{i-k_i} and β_{ik_i}, where $1 \leq k_i \leq n$-1.

Lemma 2.4 After execution of ***Algorithm_Transform***, the facet inequality (14) is transformed into one in which $(\alpha, \beta) \geq 0$ with $\alpha \neq 0$, and $\beta_0 > 0$.

Proof: It is apparent that after Step I), $\beta \geq 0$, and after step II), $\alpha \geq 0$. Clearly, step III) of ***Algorithm_Transform*** ensures that $\alpha \neq 0$. Its execution indicates that $\alpha_{ijk_i} > 0$ for some $i \in M$, $1 \leq k_i \leq n$, for each $j \in N$. Hence, $\alpha \geq 0$ with $\alpha \neq 0$. For each $l \in I_{mn}$, we have that

$$\sum_{i \in M'(l)} \sum_{j \in N_{k_i}(l)} \alpha_{ijk_i} = \sum_{i \in M'(l)} \beta_{ik_i} + \beta_0. \tag{17}$$

Suppose $\beta_0 \leq 0$. Then (14) is transformed using (16a) and (16b), wherein $\Delta = -\beta_0/n$. After this transformation, $\beta_0 = 0$ and that $\alpha > 0$. Next, in (14), using (6) and (7), the y variables are substituted out. Consequently, (14) takes the form: $\alpha' z \leq 0$, where α' denotes the vector of revised α values after the substitution of y can be suitably converted to one with $\beta_0 = 0$. By definition, $\alpha' z = 0$ for each $(z, y) \in S_{mn}$, which are $mn(n\text{-}1)+m\text{-}1$ in number. This results in $\alpha'_{ijk_i}=0$ for each $i \in M$, $j \in N$, $1 \leq k_i \leq n$, suggesting that (14) is a linear combination of constraints (6) and (7) making it trivial. Hence, $\beta_0 > 0$. □

Remark 2.2 It needs to be noted that due to step II) of ***Algorithm_Transform***, either $\alpha_{ijk_i} = 0$ for at least one $j \in N$, for each $i \in M$ and $k_i = 1,..,n\text{-}1$, or that $\beta_{ik_i} = 0$. We refer to the resulting inequality (14) as *minimal*, and the z variables whose coefficients $\alpha_{ijk_i} = 0$, as *hidden* assignments. They will play an important role in the construction of Cardinality Matching inequalities alluded to later in this paper.

Definition 2.1 Let $\widehat{M} \subseteq M$ be defined as the *smallest* set of agents in M such that $\widehat{M} \cap M'(l) \neq \emptyset$ for each $l \in I_{mn}$ in which $|M'(l)| \geq 2$. That is, $\widehat{M}$ comprises of the *minimal* number of agents that are present in each of the $mn(n\text{-}1)\text{-}1$ affinely independent solutions in S_{mn} where $|M'(l)| \geq 2$.

There are then two distinct types of facets as they pertain to the construction of solutions in which $|M'(l)| \geq 2$. In one type, $|\widehat{M}| = 1$, and in the other, $|\widehat{M}| \geq 2$. The solution set illustrated in Figure 2 is an example of the former, while that in Figure 1 illustrates the latter. Note that in the example illustrated in Figure 1, $\widehat{M} = \{i_1, i_2\}$, if $M = \{i, i_1, i_2, i_3\}$. With regards to any facet (14), we will refer to agents in $\widehat{M}$ as *primary* agents, and the rest as *secondary* agents.

Proposition 2.1 Any *minimal* facet of $H(z, y)$ in which $\beta = 0$ reduces to either a) $z_{i^*j^*k_{i^*}} \geq 0$, for some $i^* \in M$, $j^* \in N$ and $1 \leq k_{i^*} \leq n\text{-}1$, or b) constraint (8) for some $i^* \in M$.

Proof: An equivalent form of the inequality $z_{i^*j^*k_{i^*}} \geq 0$ is

$$\sum_{i \in \{M-i^*\}} \sum_{k_i=1}^{n} z_{ij^*k_i} + \sum_{k_i=1}^{k_{i^*}-1} z_{i^*j^*k_i} + \sum_{k_i=k_{i^*}+1}^{n} z_{i^*j^*k_i} \leq 1, \qquad (18a)$$

which is obtained by adding $-z_{i^*j^*k_{i^*}} \leq 0$ to the equality constraint (4) associated with j^*. Another equivalent form of inequality $z_{i^*j^*k_{i^*}} \geq 0$ is obtained by executing step III) of ***Algorithm_Transform*** on (18a) with $\alpha_{max} = 1$. The resulting inequality obtained is:

$$\sum_{i \in \{M-i^*\}} \sum_{j \in N} \sum_{k_i=1}^{n} z_{ijk_i} + \sum_{j \in \{N-j^*\}} \sum_{k_i=1}^{n} z_{i^*jk_i} + \sum_{k_i=1}^{k_{i^*}-1} z_{i^*j^*k_i} + \sum_{k_i=k_{i^*}+1}^{n} z_{i^*j^*k_i} \leq n. \qquad (18b)$$

By substituting out all the y variables using (6) and (7), constraint (8) for i^* reduces to

$$\sum_{j \in N} \sum_{k_{i^*}=1}^{n-1} \frac{1}{k_{i^*}} z_{i^*jk_{i^*}} + \sum_{i \in M} \sum_{j \in N} \frac{1}{n} z_{ijn} \leq 1. \qquad (19)$$

We now proceed to show that all *minimal* facets of $H(z, y)$ with $\beta = 0$ reduce to either (18a) or (19). For such facets, the following equality:

$$\sum_{i\in M'(l)} \sum_{j\in N_{k_i}(l)} \alpha_{ijk_i} = \beta_0, \tag{20}$$

results for each $l\in I_{mn}$. In the case of the m solutions in S_{mn}: $y_{in} = 1$, $z_{ijn} = 1$ for all $j\in N$, for each $i\in M$, (20) reduces to $\sum_{j\in N} \alpha_{ijn} = \beta_0$. To identify the remaining $mn(n\text{-}1)\text{-}1$ affinely independent solutions in S_{mn}, in which $|M'(l)| \geq 2$, we consider two dichotomous cases: a) $|\widehat{M}| \geq 2$, and b) $|\widehat{M}| = 1$.

We begin with the case of $|\widehat{M}| \geq 2$. Consider first those solutions $(z, y)\in S_{mn}$ in which $M'(l) = \{i_1, i_2\}\subseteq\widehat{M}$, $k_{i_1} = 1$, $k_{i_2} = n-1$ and j_1 is assigned to i_1. Assume for the moment that $k_{i^*} \neq 1$ and $k_{i^*} \neq n\text{-}1$. Thus, if i_1 is replaced by another primary agent $i\in\widehat{M}$ to which j_1 is assigned to, then (20) again holds. The same holds true if i_1 is replaced by a secondary agent $i\in\{M\text{-}\widehat{M}\}$. Thus, $\alpha_{i_1 j1} = \alpha_{i_2 j1} = \cdots = \alpha_{i_m j1} = \alpha_{j1}$. If $k_{i^*} = 1$, then the above holds for all $i \neq i^*$.

Using induction, it can be shown that, except for $\alpha_{i^* j^* k_{i^*}}$, all coefficients in $\alpha(j)$ are equal, for each $j\in N$. This is accomplished by sequentially switching between solutions that exclude $z_{i^* j^* k_{i^*}} = 1$. Consider a $(z, y)\in S_{mn}$ in which $M'(l) = \{i_1, i_2\}$, with $k_{i_1} = 2$, $k_{i_2} = n-2$ and $i_2\in\widehat{M}$. Further, $N_{k_{i_1}}(l)= \{j_1, j_2\}$. Then, (20) holds for both when $i_1\in\widehat{M}$ and $i_1\notin\widehat{M}$. Let i_1 be replaced by i_3 and i_4 with $k_{i_3} = k_{i_4} = 1$, and j_1 and j_2 assigned to i_3 and i_4, respectively. Regardless of whether i_3 and i_4 are primary agents or not, (20) holds. Therefore,

$$\alpha_{i_1 j_1 2} + \alpha_{i_1 j_2 2} = \alpha_{j_1 1} + \alpha_{j_2 1}. \tag{21}$$

If in (21), j_1 is replaced by j_3, then

$$\alpha_{i_1 j_3 2} + \alpha_{i_1 j_2 2} = \alpha_{j_3 1} + \alpha_{j_2 1}. \tag{22}$$

Similarly, with j_3 replacing j_2 in (21), we obtain

$$\alpha_{i_1 j_1 2} + \alpha_{i_1 j_3 2} = \alpha_{j_1 1} + \alpha_{j_3 1}. \tag{23}$$

If $(\alpha_{i_1 j_1 2}\text{-}\alpha_{j_1 1}) = \Delta \neq 0$, then from (21), $(\alpha_{i_1 j_2 2}\text{-}\alpha_{j_2 1}) = \text{-}\Delta$. From (22), one obtains $(\alpha_{i_1 j_3 2}\text{-}\alpha_{j_3 1}) = \text{-}(\alpha_{i_1 j_2 2}\text{-}\alpha_{j_2 1}) = \Delta$. However, from (23), a contradiction results with $(\alpha_{i_1 j_1 2}\text{-}\alpha_{j_1 1}) = \text{-}(\alpha_{i_1 j_3 2}\text{-}\alpha_{j_3 1}) = \text{-}\Delta$, which is only resolved when $\Delta = 0$. Therefore, $\alpha_{ij2} = \alpha_{j1}$ for each $j\in N$. Now assume that $\alpha_{jk_i} = \alpha_{jk_i-1}$, for all $j\in N$ for some $k_i \geq 2$. Consider a $l\in I_{mn}$ with $M'(l) = \{i_1, i_2\}$, $k_{i_1} = k_i + 1$, $k_{i_2} = n - k_i - 1$, satisfying (20). This solution is perturbed by replacing i_1 with i_3 and i_4, with $k_{i_3} = k_i$ and $k_{i_4} = 1$ with a $j_1\in N_{k_i+1}(l)$ now assigned to i_4. Since $\alpha_{jk_i} = \alpha_{jk_i-1}$ for all $j\in N$, for some $k_i \geq 2$, applying it recursively, one can conclude that $\alpha_{jk_i} = \alpha_{j1}$ for all $j\in N$. Therefore,

$$\sum_{j\in\{N_{k_i+1}(l)-j_1\}} \alpha_{jk_i+1} + \alpha_{j_1 k_i+1} = \sum_{j\in\{N_{k_i+1}(l)-j_1\}} \alpha_{j1} + \alpha_{j_1 1}. \tag{24}$$

Alternatively,

$$\alpha_{j_1 k_i+1} - \alpha_{j_1 1} = -\sum_{j\in\left\{N_{k_i+1}(l)-j_1\right\}} (\alpha_{j k_i+1} - \alpha_{j1}). \tag{25}$$

Now assume that, $\alpha_{j_1 k_i+1} - \alpha_{j_1 1} = \Delta \neq 0$. A $j_2 \in \{N\text{-}N_{k_i+1}(l)\}$ is selected and exchanged with j_1, i.e., after the exchange $j_2 \in N_{k_i+1}(l)$ while $j_1 \in \{N\text{-}N_{k_i+1}(l)\}$. With this exchange, (20) still holds and therefore the right-hand-side of (25) remains unchanged, i.e $\alpha_{j_2 k_i+1} - \alpha_{j_2 1} = \Delta$. It follows that $\sum_{j\in\left\{N_{k_i+1}(l)-j_2\right\}} (\alpha_{j k_i+1} - \alpha_{j1}) = -\Delta$. Therefore, there must be a $j_3 \in \{N_{k_i+1}(l) - j_2\}$, for which $\alpha_{j_3 k_i+1} - \alpha_{j_3 1} = \Delta_3 < 0$ if $\Delta > 0$, and $\Delta_3 > 0$ if $\Delta < 0$. Let j_1 and j_3 be exchanged wherein $j_1 \in N_{k_i+1}(l)$ and $j_3 \in \{N\text{-}N_{k_i+1}(l)\}$. Consequently, $-\sum_{j\in\left\{N_{k_i+1}(l)-j_2\right\}} (\alpha_{j k_i+1} - \alpha_{j1})$ changes by $(\Delta\text{-}\Delta_3) \neq 0$. With this new exchange, $\alpha_{j_2 k_i+1} - \alpha_{j_2 1}$ also changes by $(\Delta\text{-}\Delta_3) \neq 0$, a contradiction, which is only resolved if $\Delta = \Delta_3 = 0$. Thus, for each $j \in \{N\text{-}j^*\}$, $\alpha_{j k_i} = \alpha_j$, for $k_i = 1,.., n\text{-}1$. When $j = j^*$, $\alpha_{j^* k_i} = \alpha_{j^*}$ for each $k_i \neq k_{i^*}$, and $\alpha_{i^* j^* k_{i^*}} < \alpha_{j^*}$. With regards to coefficients α_{ijn}, using (6) are modified as follows. First determine, $\Delta_i = (\alpha_j - \alpha_{ijn})$, for each $i \in M$. Due to (6), coefficients α_{ijn} and β_{in} are modified as, $\alpha_{ijn} \to \alpha_{ijn} + \Delta_i$ and $\beta_{in} \to \beta_{in} + \Delta_i$, respectively. This ensures that $\alpha_{ijn} = \alpha_j$ for each $j \in N$. Let, By setting $\Delta = (\alpha_{max}\text{-}\alpha_j)$ where $\alpha_{max} = Max\ \{\alpha_j \mid j \in N\}$, $\alpha(j)$ and β_0 are updated as per (16a) and (16b) for each j. As a result, an equivalent inequality in which α_j is the same for all j, except for $\alpha_{i^* j^* k_{i^*}}$ is obtained. Next, by setting $\Delta = \text{-}\alpha_{i^* j^* k_{i^*}}$ and updating $\alpha(j)$ and β_0 as per (16a) and (16b) for each j, an equality which is a multiple of (18a) is obtained.

Now consider the case of $|\widehat{M}| = 1$ with $\widehat{M} = \{\hat{\imath}\}$. Then, for each $l \in I_{mn}$ since $|M'(l)| \geq 2$, $M'(l)$ must contain at least one secondary agent. Consider a $(z, y) \in S_{mn}$ consisting of $M'(l) = \{i_1, \hat{\imath}\}$, $k_{i_1} = 1$, $k_{\hat{\imath}} = n - 1$, with j assigned to i_1 (a secondary agent). By keeping the assignment to $\hat{\imath}$ fixed and reassigning j to another agent $i \in \{M\text{-}M'(l)\}$, (20) holds. Hence, $\alpha_{i_1 j1} = \alpha_{j1}$ for each $i_1 \in \{M\text{-}\widehat{M}\}$. Now consider a $(z, y) \in S_{mn}$ in which $M'(l) = \{i_1, \hat{\imath}\}$, $k_{i_1} = 2$, $k_{\hat{\imath}} = n - 2$. Again, by keeping the assignment to $\hat{\imath}$ fixed, and varying the assignment to i_1, as described above in (21), (22) and (23), it is easy to show that $\alpha_{i_1 j2} = \alpha_{j1}$ for each $j \in N$ and $i_1 \in \{M\text{-}\widehat{M}\}$. Extending this argument inductively from k_{i_1} to $k_{i_1} + 1$ using the logic of (24) and (25), $\alpha_{j k_i} = \alpha_j$ for each $j \in N$ and $i \in \{M\text{-}\hat{\imath}\}$.

Consider again a $l \in I_{mn}$ in which $M'(l) = \{i_1, \hat{\imath}\}$, with $j_1 \in N_{k_{i_1}}(l)$ and $j_2 \in \{N\text{-}N_{k_{i_1}}(l)\} = N_{k_{\hat{\imath}}}(l)$ for which (20) holds. The feasible solution in which the assignments of j_1 and j_2 are switched also belongs to S_{mn}. Consequently

$$\alpha_{j_1} + \alpha_{\hat{\imath} j_2 k_{\hat{\imath}}} = \alpha_{j_2} + \alpha_{\hat{\imath} j_1 k_{\hat{\imath}}}, \tag{26}$$

which alternately can be stated as

$$\alpha_{\hat{\imath} j_1 k_{\hat{\imath}}} - \alpha_{\hat{\imath} j_2 k_{\hat{\imath}}} = \alpha_{j_1} - \alpha_{j_2}, \tag{27}$$

for every pair $\{j_1, j_2\} \in N$ and every $1 \leq k_{\hat{\imath}} \leq n\text{-}1$. Consider another perturbation of the solution, $M'(l) = \{i_1, \hat{\imath}\}$, in which the assignment of jobs in $N_{k_{i_1}}(l)$ to i_1 is fixed, but jobs in $N_{k_{\hat{\imath}}}(l)$ are now assigned to another $i_2 \neq \hat{\imath}$. Since (20) holds only for solutions that involve $\hat{\imath}$, it follows that

$$\sum_{j \in N_{k_{\hat{\imath}}}(l)} \alpha_j < \sum_{j \in N_{k_{\hat{\imath}}}(l)} \alpha_{\hat{\imath} j k_{\hat{\imath}}}, \tag{28}$$

for every selection of $N_{k_{\hat{\imath}}}(l) \subset N$. Naturally, $\alpha_j < \alpha_{\hat{\imath} j k_{\hat{\imath}}}$ for every $j \in N$. Since (28) holds for every $1 \le k_{\hat{\imath}} \le n$-1, $\alpha_j < \alpha_{\hat{\imath} j k_{\hat{\imath}}}$ for each $1 \le k_{\hat{\imath}} \le n$-1. As a first step, the α coefficients are modified as follows. For each $j \in N$, $\Delta \rightarrow -\alpha_j$ and (16a) and (16b) is executed. Consequently, after the update, $\alpha_{ijk_i} = 0$ for each $i \in \{M-\hat{\imath}\}$, $1 \le k_i \le n$. Since $\alpha_j < \alpha_{\hat{\imath} j k_{\hat{\imath}}}$, $\alpha_{\hat{\imath} j k_{\hat{\imath}}} > 0$ after the update. Further, due to (27), after the update, $\alpha_{\hat{\imath} j k_{\hat{\imath}}} = \alpha_{\hat{\imath} k_{\hat{\imath}}}$ for each $j \in N$, and $1 \le k_{\hat{\imath}} \le n$-1. As well, $\beta_0 > 0$ after the update. In order to satisfy (14) as an equality, $k_{\hat{\imath}} \alpha_{\hat{\imath} k_{\hat{\imath}}} = \beta_0$. Therefore, $\alpha_{\hat{\imath} k_{\hat{\imath}}} = \beta_0 / k_{\hat{\imath}}$, and the resulting inequality is a multiple of (19). □

The import of Proposition 2.1 is that in any minimal, *non-trivial* facet of $H(z, y)$, $(\alpha, \beta) \ge 0$ with $\beta \ne 0$. It then naturally follows that if for any $i \in M$ and $1 \le k_i \le n$-1, $\beta_{ik_i} > 0$, then $\alpha^{i-k_i} \ne 0$. Otherwise, (14) cannot be a *non-trivial* facet of $H(z, y)$.

Definition 2.2 For each $i \in M$, $1 \le k_i \le n$-1, $N^{min}_{i-k_i} = \{j' \in N | \ \alpha_{ij'k_i} = Min\{\alpha_{ijk_i} | j \in N\}\}$.

Lemma 2.5 For any $(z, y) \in S_{mn}$ associated with a minimal, non-trivial facet of $H(z, y)$, $\bigcap_{i \in M'(l)} N^{min}_{i-k_i} \ne \phi$, in which $1 \le k_i \le n$-1 for each $i \in M'(l)$.

Proof: The proof of the above is constructed by contradiction. Assume that for some $l \in I_{mn}$, $\bigcap_{i \in M'(l)} N^{min}_{i-k_i} = \phi$. Let such a feasible solution be denoted as $(z, y)^1$. There must exist a pair $\{i_1, i_2\} \subseteq M'(l)$ in $(z, y)^1$ for which $N^{min}_{i_1-k_{i_1}} \cap N^{min}_{i_2-k_{i_2}} = \phi$. There are three possibilities with regards to the composition of $N_{k_{i_1}}(l)$ and $N_{k_{i_2}}(l)$. One possibility is that there exists a $j_{i_1} \in N^{min}_{i_1-k_{i_1}} \cap N_{k_{i_1}}(l)$, as well as a $j_{i_2} \in N^{min}_{i_2-k_{i_2}} \cap N_{k_{i_2}}(l)$. By assumption, $j_{i_1} \notin N^{min}_{i_2-k_{i_2}}$ and $j_{i_2} \notin N^{min}_{i_1-k_{i_1}}$. Consequently, $\alpha_{i_1 j_{i_1} k_{i_1}} < \alpha_{i_1 j_{i_2} k_{i_1}}$ and $\alpha_{i_2 j_{i_2} k_{i_2}} < \alpha_{i_2 j_{i_1} k_{i_2}}$. Therefore, upon switching the assignments of j_{i_1} and j_{i_2}, (14) is violated, contradicting the fact that (14) is valid.

A second possibility is that while a $j_{i_1} \in N^{min}_{i_1-k_{i_1}} \cap N_{k_{i_1}}(l)$ exists, $N^{min}_{i_2-k_{i_2}} \cap N_{k_{i_2}}(l) = \phi$. Suppose that a $j_{i_2} \in N^{min}_{i_2-k_{i_2}}$ is assigned to $i_3 \in \{M'(l)\text{-}i_1\text{-}i_2\}$. We now consider the exchanging the assignment of j_{i_2} with each $j \in N_{k_{i_2}}(l)$. Since (14) is valid, $(\alpha_{i_3 j_{i_2} k_{i_3}} - \alpha_{i_2 j_{i_2} k_{i_2}}) \ge (\alpha_{i_3 j k_{i_3}} - \alpha_{i_2 j k_{i_2}})$ for each $j \in N_{k_{i_2}}(l)$. For a $j^* \in N_{k_{i_2}}(l)$, let $(\alpha_{i_3 j^* k_{i_3}} - \alpha_{i_2 j^* k_{i_2}}) = Max_{j \in N_{k_{i_2}}(l)}\{(\alpha_{i_3 j k_{i_3}} - \alpha_{i_2 j k_{i_2}})\}$. If $(\alpha_{i_3 j^* k_{i_3}} - \alpha_{i_2 j^* k_{i_2}}) = (\alpha_{i_3 j_{i_2} k_{i_3}} - \alpha_{i_2 j_{i_2} k_{i_2}})$, then upon exchanging the assignment of j_{i_2} and j^*, another feasible solution $(z, y)^2 \in S_{mn}$ is obtained. However, exchanging the assignment of j_{i_1} and j_{i_2} in $(z, y)^2$ results in (14) being violated. Hence, $(z, y)^2 \notin S_{mn}$. In fact, all feasible solutions obtained by exchanging the assignments of j_{i_2} and j in $(z, y)^1$ for each $j \in N_{k_{i_2}}(l)$ do not belong to S_{mn}. Similarly, in $(z, y)^2$, all exchanges of assignments between $j_1 \in \{N_{k_{i_1}}(l)\text{-}j_{i_1}\}$ and $j_2 \in \{N_{k_{i_2}}(l)\text{-}j_{i_2}\}$ result in feasible solutions that do not belong to S_{mn}. The same is true for all exchanges of assignments between $j_1 \in \{N_{k_{i_1}}(l)\text{-}j_{i_1}\}$ and $j \in N_{k_i}(l)$ for $i \in \{M'(l)\text{-}i_1\text{-}i_2\}$ in $(z, y)^2$. Thus, all feasible solutions in which j_{i_2} is assigned to i_2 strictly satisfy (14). This then violates the assumption that (14) is a facet. A final possibility is that in $(z, y)^1$, $N^{min}_{i_1-k_{i_1}} \cap N_{k_{i_1}}(l) = \phi$ and $N^{min}_{i_2-k_{i_2}} \cap N_{k_{i_2}}(l) = \phi$. Consider a $j_{i_1} \in N^{min}_{i_1-k_{i_1}}$ that is assigned to $i_3 \in \{M'(l)\text{-}i_1\text{-}i_2\}$. Suppose that by exchanging the assignment of j_{i_1} with a

$j\in N_{k_{i_1}}(l)$, a solution that belongs to S_{mn} is obtained, then this corresponds to the second possibility described above. This completes the proof. □

What is immediately clear from Lemma 2.5 is that in any $(z, y)\in S_{mn}$ associated with a minimal, non-trivial facet of $H(z, y)$, for at least one $j\in\{\bigcap_{i\in M'(l)} N_{i-k_i}^{min}\}$, $j\in N_{k_i}(l)$, for some $i\in M'(l)$. In any $l\in M'(l)$, let a $\hat{\jmath}\in\{\bigcap_{i\in M'(l)} N_{i-k_i}^{min}\}$ be assigned to $\hat{\imath}\in M'(l)$. Since (14) is a facet, for all $j'\in\{N| j\notin N_{i-k_i}^{min}$ for any $i\in M'(l)\}$ and which is currently assigned to $i'\in\{M'(l)\text{-}\hat{\imath}\}$ with a cardinality $k_{i'}$, the following holds:

$$\Delta_{\hat{\imath}} = \alpha_{i'j'k_{i'}} - \alpha_{\hat{\imath}j'k_{\hat{\imath}}} = \alpha_{i'\hat{\jmath}k_{i'}} - \alpha_{\hat{\imath}\hat{\jmath}k_{\hat{\imath}}}. \tag{29}$$

The same holds if j' in (29) is replaced by $j\in\{\bigcap_{i\in M'(l)} N_{i-k_i}^{min}\text{-}\hat{\jmath}\}$.

Corollary 2.1 For some $l\in I_{mn}$ associated with a minimal, non-trivial facet of $H(z, y)$ and a $j\in N_{k_i}(l)$, if $j\in N_{i-k_i}^{min}$ then $j\in\{\bigcap_{i\in M'(l)} N_{i-k_i}^{min}\}$.

Proof: To prove, assume the converse, i.e., for some $l\in I_{mn}$, there exists a $j\in N_{k_i}(l)$ such that $j\in N_{i-k_i}^{min}$, but $j\notin\{\bigcap_{i\in M'(l)} N_{i-k_i}^{min}\}$. Therefore, there must be an $i'\in M'(l)$ for which $j\notin N_{i'-k_{i'}}^{min}$. Suppose that a $\hat{\jmath}\in\{\bigcap_{i\in M'(l)} N_{i-k_i}^{min}\}$ is assigned to i'. Then, if the assignments of j and $\hat{\jmath}$ are exchanged, i.e., j assigned to i' and $\hat{\jmath}$ assigned to i, results in a violation of (14). That is because $(\alpha_{ij'k_i} - \alpha_{\hat{\imath}j'k_{\hat{\imath}}}) > (\alpha_{i\hat{\jmath}k_i} - \alpha_{\hat{\imath}\hat{\jmath}k_{\hat{\imath}}})$, unlike in (29). This contradicts the fact that (14) is valid. Suppose that no $\hat{\jmath}\in\{\bigcap_{i\in M'(l)} N_{i-k_i}^{min}\}$ is assigned to i'. Rather, $\hat{\jmath}\in\{\bigcap_{i\in M'(l)} N_{i-k_i}^{min}\}$ is assigned to $i''\in\{M'(l)\text{-}\, i'\}$. By exchanging the assignment of $\hat{\jmath}$ with that of a $j''\in N_{k_i}(l)$, the resulting solution belongs to S_{mn}, due to (29). With this perturbed solution, exchanging the assignment of j and $\hat{\jmath}$ results in (14) being violated, a contradiction. □

Lemma 2.5 and Corollary 2.1 both together suggests that in any $l\in I_{mn}$ associated with a non-trivial facet of $H(z, y)$, each $j\in N_{k_i}(l)$ of $i\in M'(l)$, is such that either $j\in\{\bigcap_{i\in M'(l)} N_{i-k_i}^{min}\}$, or $j\notin N_{i-k_i}^{min}$ for each $i\in M'(l)$.

Proposition 2.2 Let $\widehat{M}\subseteq M$ denote a set associated with a minimal facet of $H(z, y)$ as per in Definition 2.1. Then, for all minimal, *non-trivial* facets of $H(z, y)$, $|\widehat{M}| \geq 2$.

Proof: We proceed to show that all minimal facets of $H(z, y)$ whose $|\widehat{M}| = 1$ reduce to (19), a trivial facet. Let $\widehat{M} = \{\hat{\imath}\}$. Since $|M'(l)| \geq 2$, each $(z, y)\in S_{mn}$ consists of at least one secondary agent. Consider a $l\in I_{mn}$ in which $M'(l) = \{\hat{\imath}, i_2\}$, $1 \leq k_{\hat{\imath}} \leq n\text{-}1$ and $k_{i_2} = n\text{-}k_{\hat{\imath}}$. Therefore,

$$\sum_{j\in N_{k_{\hat{\imath}}}(l)} \alpha_{\hat{\imath}jk_{\hat{\imath}}} + \sum_{j\in N_{k_{i_2}}(l)} \alpha_{i_2jk_{i_2}} = \beta_{\hat{\imath}k_{\hat{\imath}}} + \beta_{i_2k_{i_2}} + \beta_0. \tag{30}$$

It is clear from Lemma 2.5 that in $M'(l)$, there exists a $j_{\hat{\imath}}\in N_{\hat{\imath}-k_{\hat{\imath}}}^{min}\cap N_{i_2-k_{i_2}}^{min}$. Further, for (14) to be a facet, n independent selections of $N_{k_{i_2}}$ from N are required that satisfy (30). Therefore, by exchanging the assignment of $j_{\hat{\imath}}$ and each $j\in\{N\text{-}j_{\hat{\imath}}\}$ between $\hat{\imath}$ and i_2, (30) must hold. Thus, substituting i_2 for i' and j for j', (29) holds, i.e., $\Delta_{\hat{\imath}} = \alpha_{\hat{\imath}jk_{\hat{\imath}}} - \alpha_{i_2jk_{i_2}}$ for each $j\in\{N\text{-}j_{\hat{\imath}}\}$.

For each secondary agent i_2, $\Delta = \Delta_{\hat{\imath}}$ as determined in (29) is used to update the coefficients $\alpha_{i_2 j k_{i_2}}$ and $\beta_{i_2 k_{i_2}}$ as specified (15a) and (15b) for each $j \in \{N\text{-}j_{\hat{\imath}}\}$. Consequently, $\alpha_{i_2 j k_{i_2}} = \alpha_{\hat{\imath} j k_{\hat{\imath}}}$, for each $i_2 \in \{M\text{-}\hat{\imath}\}$ and each $j \in N$. Next, let i_2 be replaced by two secondary agents i_3 and i_4, with respective cardinalities k_{i_3} and k_{i_4} such that $k_{i_3}+k_{i_4} = k_{i_2}$. The coefficients $\alpha_{i_3 j k_{i_3}}$, $\beta_{i_3 k_{i_3}}$, $\alpha_{i_4 j k_{i_4}}$ and $\beta_{i_4 k_{i_4}}$ are updated using (15a) and (15b) for each $j \in \{N\text{-}j_{\hat{\imath}}\}$ the same way as with i_2. Since the choice of $k_{\hat{\imath}}$, k_{i_2} and k_{i_3} is arbitrary, after the updates, $\alpha_{i_2 j k_{i_2}} = \alpha_j$ for each $j \in N$, $1 \leq k_{i_2} \leq n$-1. As well, $\alpha_{\hat{\imath} j k_{\hat{\imath}}} = \alpha_j$ for each $j \in N$, $1 \leq k_{\hat{\imath}} \leq n$-1. The choice of i_2 being arbitrary, $\beta_{i_2 k_{i_2}} = \beta_{k_{i_2}}$, for each $i_2 \neq \hat{\imath}$, which holds true for each $1 \leq k_{i_2} \leq n$-1. Another consequence of this transformation is that the l-h-s of (30) reduces to $\sum_{j \in N} \alpha_j$ for each $l \in I_{mn}$. To complete the transformation, $\alpha_{ijn} \rightarrow \alpha_j$, for each $i \in M$ and $\beta_{in} \rightarrow \sum_{j \in N} \alpha_j - \beta_0$, for each $i \in M$. We next define $\alpha_0 = (\sum_{j \in N} \alpha_j)/n$, and execute transformation (16a) and (16b) for each $j \in N$, with $\Delta = (\alpha_0 - \alpha_j)$. As a result, $\alpha_{ijk_i} = \alpha_0$ for each $i \in M$, $1 \leq k_i \leq n$-1. Since $n\alpha_0 = \sum_{j \in N} \alpha_j$, β_0 remains unchanged. Since the l-h-s of (30) is equal to $n\alpha_0$ for each $l \in I_{mn}$, $\beta_{\hat{\imath} k_{\hat{\imath}}} + \beta_{i_2 k_{i_2}} = \beta_{\hat{\imath} k_{\hat{\imath}}+1} + \beta_{i_2 k_{i_2}-1}$, for each $i_2 \in \{M\text{-}\hat{\imath}\}$ and $2 \leq k_{i_2} \leq n$-1, given that $k_{\hat{\imath}}+k_{i_2} = n$. Stated in another way, $(\beta_{\hat{\imath} k_{\hat{\imath}}+1} - \beta_{\hat{\imath} k_{\hat{\imath}}}) = (\beta_{i_2 k_{i_2}} - \beta_{i_2 k_{i_2}-1})$ for $2 \leq k_{i_2} \leq n$-1. If i_2 is replaced by $\{i_3, i_4\} \in \{M\text{-}\hat{\imath}\}$, in which $k_{i_3} = k_{i_2} - 1$, and $k_{i_4} = 1$, then such a solution belongs to S_{mn} as well. Thus, $\beta_{k_{i_2}} = \beta_{k_{i_2}-1} + \beta_1$, which holds for all $2 \leq k_{i_2} \leq n$-1, which in turn implies that $\beta_{k_{i_2}} = k_{i_2}\beta_1$. Similarly, $(\beta_{\hat{\imath} k_{\hat{\imath}}+1} - \beta_{\hat{\imath} k_{\hat{\imath}}}) = \beta_1$ for $1 \leq k_{\hat{\imath}} \leq n$-2, implying that $\beta_{\hat{\imath} k_{\hat{\imath}}} = (k_{\hat{\imath}} - 1)\beta_1 + \beta_{\hat{\imath}1}$ for $1 \leq k_{\hat{\imath}} \leq n$-1.

By definition, for each $l \in I_{mn}$, $\hat{\imath}$ necessarily belongs to $M'(l)$. Thus, if $\hat{\imath}$ is replaced by $i_1 \in \{M\text{-}\hat{\imath}\text{-}i_2\}$ in $M'(l)$, then (30) becomes a strict less-than inequality. This suggests that $\beta_{\hat{\imath} k_i} < \beta_{i k_i}$ for each $i \in \{M\text{-}\hat{\imath}\}$, $1 \leq k_i \leq n$-1. Therefore, $\beta_{\hat{\imath}1} < \beta_1$. That being the case, (30) can be rewritten as:

$$n\alpha_0 = (k_{\hat{\imath}} - 1)\beta_1 + \beta_{\hat{\imath}1} + k_{i_2}\beta_1 + \beta_0. \tag{31}$$

Since n is integer and $(k_{\hat{\imath}}+k_{i_2}) = n$, it follows that $\alpha_0 = \beta_1 = \beta_0$, while $\beta_{\hat{\imath}1} = 0$ satisfies (31). Using (6) and (7) to substitute out the y variables results in (14) to be a multiple of (19), a trivial facet. □

Definition 2.3 For a given $l \in I_{mn}$, let y_{ik_i}=1 for an $i \in M'(l)$, along with $z_{ijk_i} = 1$, for $j \in N_{k_i}(l)$. Then *Co-l*(i-k_i) denotes the complementary part of this solution, i.e. $y_{i'k_{i'}}$=1 and $z_{i'jk_{i'}} = 1$ for each $j \in N_{k_{i'}}(l)$, $i' \in \{M'(l)\text{-}i\}$.

Lemma 2.6 In any minimal, non-trivial facet of $H(z, y)$, for each $1 \leq k_i \leq n$-1, $\beta_{i_1 k_i} = \beta_{i_2 k_i}$ and $\alpha^{i_1 - k_i} = \alpha^{i_2 - k_i}$ for all pairs $\{i_1, i_2\} \in \{M\text{-}\widehat{M}\}$.

Proof: Consider solutions $(z, y) \in S_{mn}$ in which $M'(l) = \widehat{W} \cup i_1$, with $\widehat{W} \subseteq \widehat{M}$ and $i_1 \notin \widehat{M}$. By virtue of $(z, y) \in S_{mn}$,

$$\sum_{j \in N_{k_{i_1}}(l)} \alpha_{i_1 j k_{i_1}} - \beta_{i_1 k_{i_1}} = \sum_{\hat{\imath} \in \widehat{W}} \left(\beta_{\hat{\imath} k_{\hat{\imath}}} - \sum_{j \in N_{k_{\hat{\imath}}}(l)} \alpha_{\hat{\imath} j k_{\hat{\imath}}} \right) + \beta_0. \tag{32}$$

In $M'(l)$, if i_1 is replaced by another $i \notin \widehat{M}$ with $N_{k_i}(l) = N_{k_{i_1}}(l)$, (32) will hold for each $i \in \{M\text{-}\widehat{M}\text{-}i_1\}$. Since (14) is minimal, $\sum_{j \in N_{k_i}(l)} \alpha_{ijk_i} = \sum_{j \in N_{k_{i_1}}(l)} \alpha_{i_1 j k_{i_1}}$. Therefore, for each pair $\{i_1, i_2\} \in \{M\text{-}\widehat{M}\}$ and $1 \leq k_i \leq$

n-1, $\beta_{i_1 k_i} = \beta_{i_2 k_i}$. It is clear from (32) that for any pair $\{i_1, i_2\} \in \{M\text{-}\widehat{M}\}$ and $1 \le k_{i_1} \le n$-1, $Co\text{-}S(i_1\text{-}k_{i_1})$ = $Co\text{-}S(i_2\text{-}k_{i_1})$. Therefore, for each $1 \le k_i \le n$-1, α^{i-k_i} is equal for all $i \in \{M\text{-}\widehat{M}\}$. □

It needs to be noted here that unlike in the case of secondary agents, for any pair $\{\hat{\imath}_1, \hat{\imath}_2\} \in \widehat{M}$ and $1 \le k_{\hat{\imath}_1} \le n$-1, $Co\text{-}S(\hat{\imath}_1\text{-}k_{\hat{\imath}_1})$ need not equal $Co\text{-}S(\hat{\imath}_2\text{-}k_{\hat{\imath}_1})$. Therefore, $\alpha^{\hat{\imath}_1 - k_{\hat{\imath}_1}}$ need not equal $\alpha^{\hat{\imath}_2 - k_{\hat{\imath}_2}}$, where $k_{\hat{\imath}_1} = k_{\hat{\imath}_2}$.

Definition 2.4 $\alpha min_{ik_i} = Min_{j\in N}\ \{\alpha_{ijk_i}|$ for each $i \in M$, $1 \le k_i \le n$-1}.

It is clear from Lemma 2.6 that in any minimal, non-trivial facet of $H(z, y)$, since α^{i-k_i} is equal for all $i \in \{M\text{-}\widehat{M}\}$, $\alpha min_{i_1 k_i} = \alpha min_{i_2 k_i}$ for each pair $\{i_1, i_2\} \in \{M\text{-}\widehat{M}\}$ and $1 \le k_i \le n$-1. We proceed to show the same for each pair, $\{\hat{\imath}_1, \hat{\imath}_2\} \in \widehat{M}$.

Lemma 2.7 In any minimal, non-trivial facet of $H(z, y)$, $\alpha min_{\hat{\imath}_1 k_{\hat{\imath}}} = \alpha min_{\hat{\imath}_2 k_{\hat{\imath}}}$ for each pair $\{\hat{\imath}_1, \hat{\imath}_2\} \in \widehat{M}$, $1 \le k_{\hat{\imath}} \le n$-1.

Proof: We begin with the case for $k_{\hat{\imath}} = 1$. Consider a $l \in I_{mn}$ in which $M'(l) = \{\hat{\imath}_1, i_2\}$ with $\hat{\imath}_1 \in \widehat{M}$, $k_{\hat{\imath}_1}$=1, $i_2 \in \{M\text{-}\widehat{M}\}$ and $k_{i_2} = n$-1. If $\hat{\imath}_1$ is replaced by a $\hat{\imath} \in \{\widehat{M}\text{-}\hat{\imath}_1\}$ in $M'(l)$ with $k_{\hat{\imath}} = 1$, the resulting solution also belongs to S_{mn}. This holds for any $\hat{\imath} \in \widehat{M}$. Let $j_{\hat{\imath}(min)} \in \{N^{min}_{\hat{\imath}-k_{\hat{\imath}}} \cap N^{min}_{i_2-k_{i_2}}\}$. Then, due to Lemma 2.5, a feasible solution in which $j_{\hat{\imath}(min)}$ is assigned to $\hat{\imath}$, and the rest to i_2 belongs to S_{mn}. Therefore, $(\beta_{\hat{\imath}1} - \alpha_{\hat{\imath} j_{\hat{\imath}(min)} 1})$ is equal for all $\hat{\imath} \in \widehat{M}$. Since (14) is minimal, $\alpha_{\hat{\imath} j_{\hat{\imath}(min)} 1} = \alpha min_{\hat{\imath}1}$ is equal for all $\hat{\imath} \in \widehat{M}$. As well, $\beta_{\hat{\imath}1}$ is equal for all $\hat{\imath} \in \widehat{M}$.

Next, consider a $l \in I_{mn}$ in which $M'(l) = \{\widehat{W}, \hat{\imath}_1\}$. Here, for some $\hat{\imath}_1 \in \widehat{M}$ and $\widehat{W} \subseteq \{\widehat{M}\text{-}\hat{\imath}_1\}$, $k_{\hat{\imath}} = 1$ for each $\hat{\imath} \in \widehat{W}$ and $k_{\hat{\imath}_1} = n\text{-}|\widehat{W}|$. Due to Lemma 2.5, there exists a $j_{\hat{\imath}(min)} \in \bigcap_{\hat{\imath}\in M'(l)} N^{min}_{\hat{\imath}-k_{\hat{\imath}}}$. Since $(\beta_{\hat{\imath}1} - \alpha_{\hat{\imath} j_{\hat{\imath}(min)} 1})$ is equal for all $\hat{\imath} \in \widehat{M}$, replacing $\hat{\imath}_1$ with $\hat{\imath}_2 \in \{\widehat{M}\text{-}\hat{\imath}_1\}$ and reconstituting $\widehat{W}$, with $k_{\hat{\imath}_2} = n\text{-}|\widehat{W}|$ and $k_{\hat{\imath}} = 1$ for each $\hat{\imath} \in \widehat{W}$, the replaced solution also belongs to S_{mn}. Therefore,

$$\sum_{j\in N_{k_{\hat{\imath}_1}}(l)} \alpha_{\hat{\imath}_1 j\, k_{\hat{\imath}_1}} - \beta_{\hat{\imath}_1\, k_{\hat{\imath}_1}} \;=\; \sum_{j\in N_{k_{\hat{\imath}_2}}(l)} \alpha_{\hat{\imath}_2 j\, k_{\hat{\imath}_2}} - \beta_{\hat{\imath}_1\, k_{\hat{\imath}_2}}. \tag{33}$$

Since (14) is minimal, it follows that $\alpha min_{\hat{\imath}_1 k_{\hat{\imath}}} = \alpha min_{\hat{\imath}_2 k_{\hat{\imath}}}$ for each $\{\hat{\imath}_1, \hat{\imath}_2\} \in \widehat{M}$, $n\text{-}|\widehat{M}|+1 \le k_{\hat{\imath}} \le n$-1. Finally, consider an $l \in I_{mn}$ in which $M'(l) = \{\widehat{M}, i_2\}$ with $i_2 \in \{M\text{-}\widehat{M}\}$. Here, there is a designated $\hat{\imath}_1 \in \widehat{M}$ with $1 \le k_{\hat{\imath}_1} \le n\text{-}|\widehat{M}|$, $k_{\hat{\imath}} = 1$ for each $\hat{\imath} \in \{\widehat{M}\text{-}\hat{\imath}_1\}$, and $k_{i_2} = n\text{-}k_{\hat{\imath}_1}\text{-}|\widehat{M}|+1$. As with (33), $\sum_{j\in N_{k_{\hat{\imath}_1}}(l)} \alpha_{\hat{\imath}_1 j\, k_{\hat{\imath}_1}} - \beta_{\hat{\imath}_1\, k_{\hat{\imath}_1}}$ is the same for all $\hat{\imath}_1 \in \widehat{M}$, for a given $1 \le k_{\hat{\imath}_1} \le n\text{-}|\widehat{M}|$. With (14) being minimal, $\alpha min_{\hat{\imath}_1 k_{\hat{\imath}}} = \alpha min_{\hat{\imath}_2 k_{\hat{\imath}}}$ for each $\{\hat{\imath}_1, \hat{\imath}_2\} \in \widehat{M}$, $1 \le k_{\hat{\imath}} \le n\text{-}|\widehat{M}|$. □

From the minimality of non-trivial facet (14) and Lemma 2.7, the following property is immediate.

Corollary 2.2 In any minimal, non-trivial facet of $H(z, y)$, $\beta_{\hat{\imath}_1 1} = \beta_{\hat{\imath}_2 1}$ for each pair $\{\hat{\imath}_1, \hat{\imath}_2\} \in \widehat{M}$.

Proposition 2.4 In any minimal, non-trivial facet of $H(z, y)$, $\alpha min_{ik_i} = 0$, for each $i \in M$, $1 \le k_i \le n$-1.

Proof: To prove, we first show that for each $i \in M$, $\alpha min_{ik_i} = \alpha min_i$ for all $1 \leq k_i \leq n$-1. If so, then due to Lemma 2.6 and Lemma 2.7, it follows that $\alpha min_{ik_i} = \alpha min$ for each $i \in M$, $1 \leq k_i \leq n$-1. If $\alpha min > 0$, then due to minimality of (14), $\beta = 0$, which as per Proposition 2.1 implies that (14) is trivial.

For $\hat{\imath} \in \widehat{M}$ and a $2 \leq k_{\hat{\imath}} \leq n$-1, define $\Delta_{\hat{\imath}k_{\hat{\imath}}} = \alpha min_{\hat{\imath}k_{\hat{\imath}}}$-$\alpha min_{\hat{\imath}1}$. Consider a $(z, y)^1 \in S_{mn}$ in which $M'(l_1) = \{\hat{\imath}, \hat{\imath}_1\} \subset \widehat{M}$, with $k_{\hat{\imath}_1} = 1$ and $k_{\hat{\imath}} = n$-1. Since $\alpha min_{\hat{\imath}1} = \alpha min_{\hat{\imath}_1 1}$, due to Lemma 2.5, $\Delta_{\hat{\imath}k_{\hat{\imath}}} = \alpha_{\hat{\imath}jk_{\hat{\imath}}} - \alpha_{\hat{\imath}_1 j1}$ for each $j \in N$. Consider first the case of $\Delta_{\hat{\imath}k_{\hat{\imath}}}>0$. Due to minimality of (14), $\alpha min_{\hat{\imath}k_{\hat{\imath}}}>0$ and $\beta_{\hat{\imath}k_{\hat{\imath}}} = 0$. Let $(z, y)^1$ be perturbed to obtain $(z, y)^2$ in which $M'(l_2) = \{\hat{\imath}, \hat{\imath}_1, \hat{\imath}_2\} \subseteq \widehat{M}$, with $k_{\hat{\imath}_1} = k_{\hat{\imath}_2} = 1$ and $k_{\hat{\imath}} = n$-2. Clearly, $(z, y)^2 \in S_{mn}$. Therefore, $\Delta_{\hat{\imath}k_{\hat{\imath}}-1} = \alpha min_{\hat{\imath}k_{\hat{\imath}}-1}$-$\alpha min_{\hat{\imath}1} = \alpha_{\hat{\imath}jk_{\hat{\imath}}-1} - \alpha_{\hat{\imath}_1 j1}$ for each $j \in N$. By substituting $(z, y)^1$ and $(z, y)^2$ in (14) respectively, one obtains

$$\sum_{j \in N_{k_{\hat{\imath}}}(l_1)} \alpha_{\hat{\imath}jk_{\hat{\imath}}} - \beta_{\hat{\imath}k_{\hat{\imath}}} = \sum_{j \in N_{k_{\hat{\imath}_2}}(l_2)} \alpha_{\hat{\imath}_2 j1} - \beta_{\hat{\imath}_2 1} + \sum_{j \in N_{k_{\hat{\imath}}-1}(l_2)} \alpha_{\hat{\imath}jk_{\hat{\imath}}-1} - \beta_{\hat{\imath}k_{\hat{\imath}}-1}, \tag{34}$$

for each selection of $N_{k_{\hat{\imath}}}(l_1) \subset N$ with $N_{k_{\hat{\imath}}}(l_1) = N_{k_{\hat{\imath}_1}}(l_2) \cup N_{k_{\hat{\imath}-1}}(l_2)$. From Lemma 2.7 we have that $\alpha min_{\hat{\imath}_1 1} = \alpha min_{\hat{\imath}_2 1}$. Therefore, for (34) to hold, $\alpha min_{\hat{\imath}k_{\hat{\imath}}-1} > \alpha min_{\hat{\imath}k_{\hat{\imath}}} > 0$. The above perturbation is repeated sequentially wherein the set $M'(l_k)$ is expanded recursively by adding $\hat{\imath}_k$ is added to it with $k_{\hat{\imath}_k}$=1 while reducing the cardinality of $\hat{\imath}$ by 1. As a result, $\alpha min_{\hat{\imath}k_{\hat{\imath}}-1} > \alpha min_{\hat{\imath}k_{\hat{\imath}}} > 0$ holds for n-$|\widehat{M}|$+2 $\leq k_{\hat{\imath}} \leq n$-1. For $2 \leq k_{\hat{\imath}} \leq n$-$|\widehat{M}|$+1, let $(z, y)^k \in S_{mn}$ be defined by $M'(l_k) = \widehat{M}$. Here, besides $\hat{\imath}$, $\hat{\imath}_k \in \{M'(l_k)$-$\hat{\imath}\}$ with $k_{\hat{\imath}_k} = n$-$k_{\hat{\imath}}$-$|\widehat{M}|$+2, while for each of the remaining, $\hat{\imath}_l \in \{M'(l_k)$-$\hat{\imath}_k$-$\hat{\imath}\}$, $k_{\hat{\imath}_l} = 1$. Starting with $(z, y)^k \in S_{mn}$, consider the following perturbation. A feasible solution, $(z, y)^{k+1}$ is constructed by introducing a $i_1 \in \{M$-$\widehat{M}\}$ in $M'(l_k)$ with $k_{i_1} = 1$, while the cardinality of $\hat{\imath}$ becomes $k_{\hat{\imath}}$-1. Clearly, $(z, y)^{k+1} \in S_{mn}$. Since both $(z, y)^k$ and $(z, y)^{k+1}$ belong to S_{mn},

$$\sum_{j \in N_{k_{\hat{\imath}}}(l_k)} \alpha_{\hat{\imath}jk_{\hat{\imath}}} - \beta_{\hat{\imath}k_{\hat{\imath}}} = \sum_{j \in N_{k_{i_1}}(l_{k+1})} \alpha_{i_1 j1} - \beta_{i_1 1} + \sum_{j \in N_{k_{\hat{\imath}}-1}(l_{k+1})} \alpha_{\hat{\imath}jk_{\hat{\imath}}-1} - \beta_{\hat{\imath}k_{\hat{\imath}}-1}, \tag{35}$$

for each selection of $N_{k_{\hat{\imath}}}(l_k) \subset N$. We know from the proof of Proposition 2.2 that $(\sum_{j \in N_{k_{\hat{\imath}_l}}(l_k)} \alpha_{\hat{\imath}_k j1} - \beta_{\hat{\imath}_k 1}) > (\sum_{j \in N_{k_{\hat{\imath}_l}}(l_k)} \alpha_{i_1 j1} - \beta_{i_1 1})$ for each $N_{k_{\hat{\imath}_l}}(l_k) \subset N$. Therefore, for (35) to hold, $\alpha min_{\hat{\imath}k_{\hat{\imath}}-1} > \alpha min_{\hat{\imath}k_{\hat{\imath}}} > 0$ for $2 \leq k_{\hat{\imath}} \leq n$-$|\widehat{M}|$+1. However, the fact that $\alpha min_{\hat{\imath}1} > \alpha min_{\hat{\imath}2}$ violates the initial assumption that $\alpha min_{\hat{\imath}k_{\hat{\imath}}} > \alpha min_{\hat{\imath}1}$. Hence, $\Delta_{\hat{\imath}k_{\hat{\imath}}} \leq 0$ for $\hat{\imath} \in \widehat{M}$, $2 \leq k_{\hat{\imath}} \leq n$-1.

For each $i \in \{M$-$\widehat{M}\}$ and $2 \leq k_i \leq n$-1, let $\Delta_{ik_i} = \alpha min_{ik_i}$-$\alpha min_{\hat{\imath}1}$. Using arguments, similar to that for $\Delta_{\hat{\imath}k_{\hat{\imath}}}$, $2 \leq k_{\hat{\imath}} \leq n$-1, it can be shown that $\Delta_{ik_i} \leq 0$ for $i \in \{M$-$\widehat{M}\}$, $2 \leq k_i \leq n$-1. Suppose that $\Delta_{\hat{\imath}k_{\hat{\imath}}} < 0$ for one or more $\hat{\imath} \in \widehat{M}$, $2 \leq k_{\hat{\imath}} \leq n$-1. Similarly, let $\Delta_{ik_i} < 0$ for $i \in \{M$-$\widehat{M}\}$, $2 \leq k_i \leq n$-1. Then, $\alpha min_{\hat{\imath}1} > 0$ and $\beta_{\hat{\imath}_1 1} = 0$. Further, $\alpha min_{\hat{\imath}1} \geq \alpha min_{\hat{\imath}k_{\hat{\imath}}}$ for $2 \leq k_{\hat{\imath}} \leq n$-1, and $\alpha min_{\hat{\imath}1} \geq \alpha min_{ik_i}$ for each $i \in \{M$-$\widehat{M}\}$, $2 \leq k_i \leq n$-1. In such a case, set $\Delta \rightarrow$ -$\alpha min_{\hat{\imath}1}$ and execute (16a) and (16b) for each $j \in N$. Consequently, for some $\hat{\imath} \in \widehat{M}$, $2 \leq k_{\hat{\imath}} \leq n$-1 and $i \in \{M$-$\widehat{M}\}$, $2 \leq k_i \leq n$-1, $\alpha min_{\hat{\imath}k_{\hat{\imath}}} < 0$ and $\alpha min_{ik_i} < 0$, respectively. In all such cases, (15a) and (15b) is executed by setting $\Delta \rightarrow$ -$\alpha min_{\hat{\imath}k_{\hat{\imath}}}$ and $\alpha min_{\hat{\imath}k_{\hat{\imath}}} \rightarrow \alpha min_{ik_i}$, respectively. After both these transformations, $\beta_0 > 0$. Otherwise, (14) cannot be valid. Therefore, $\alpha min_{ik_i} = 0$, for each $i \in M$, $1 \leq k_i \leq n$-1 and the proposition holds. If $\Delta_{\hat{\imath}k_{\hat{\imath}}}$=0 for each $\hat{\imath} \in \widehat{M}$, $2 \leq k_{\hat{\imath}} \leq n$-1, and Δ_{ik_i}=0 for $i \in \{M$-$\widehat{M}\}$, $2 \leq k_i \leq n$-1, then as well the proposition holds if $\alpha min_{\hat{\imath}1}$ = 0. On the other hand, if $\alpha min_{\hat{\imath}1}$ > 0, then (14) reduces to trivial facets as per Proposition 2.1. □

An immediate consequence of Proposition 2.4 is that in any non-trivial facet (14), there exists at least one $j_{ik_i}\in N$, whose potential assignment to i with cardinality $1 \leq k_i \leq n$-1, is hidden, i.e., coefficient $\alpha_{ij_{ik_i}k_i} = \alpha min_{ik_i} = 0$. What also follows from Lemma 2.5 is that in any non-trivial facet of $H(z, y)$, α_{ijk_i} = 0 for every $j\in N^{min}_{i-k_i}$, $1 \leq k_i \leq n$-1. In addition, for each $l\in I_{mn}$ associated with a non-trivial facet of $H(z, y)$, (29) applies. This suggests that $\alpha_{ijk_i} = \alpha_0 > 0$ for each $j\notin \bigcap_{i\in M'(l)} N^{min}_{i-k_i}$, $l\in I_{mn}$ associated with a non-trivial facet of $H(z, y)$. This greatly simplifies the structure of non-trivial facets of $H(z, y)$. For each $l\in I_{mn}$,

$$\sum_{i\in M'(l)} \sum_{j\in\{N_{k_i}(l)-N^{min}_{i-k_i}\}} \alpha_0 = \sum_{i\in M'(l)} \beta_{ik_i} + \beta_0. \tag{36}$$

Clearly in (36), both β_{ik_i} for each $i\in M'(l)$, and β_0 are multiples of α_0. Finally, α_{ijn} = 1, for each $i\in M$, $j\in N$ and $\beta_{in} = n-\beta_0$, after dividing by α_0. It immediately follows that all non-trivial facets of $H(z, y)$ are canonical in which all coefficients of z variables are 0-1, while $\beta_{ik_i}\geq 0$ and integer for each $i\in M$, $1 \leq k_i \leq n$. As well, $\beta_0 > 0$ and integer. Thus, the l-h-s of (14) can be viewed as a 'count' of jobs whose assignment is not hidden (i.e., those whose coefficients equal to 1), while its r-h-s can be viewed as the total cardinality provided by agents to 'match' the count of non-hidden jobs. With this, we now frame the non-trivial facet inequalities of $H(z, y)$ as p-Agent Cardinality Matching inequalities.

3.0 *p*-Agent Cardinality Matching Inequalities for (P_{zy})

The p-Agent Cardinality Matching inequalities are defined by the set of, a) $W_p \subseteq M$ agents with $|W_p| = p \geq 3$, and b) $H_q\subseteq N'\subseteq N$ jobs. In Di Francesco et. al [15], the Cardinality Matching inequalities with p = 2, has been described in some detail. To avoid repetition, we will confine ourself to discussing these inequalities with $p \geq 3$. For each $i\in W_p$, a 'partitioning' cardinality $2 \leq k(i) \leq n-p+1$ is defined which partitions the cardinalities of i, into two sets: a) $K_i^- = \{1,..., k(i)-1\}$, and b) $K_i^+ = \{k(i),…, n-1\}$. Given a fractional extreme solution $(z, y)\in LP(z, y)$, the sets defined above neatly isolates in (z, y), the fractional part. The fractional part is confined to agents $i\in W_p$ and jobs $j\in N'$. Thus, for each $i\in W_p$ and $j\in N'$, $0 < y_{ik_i} < 1$ and $0 < z_{ijk_i} < 1$ for at least one $1\leq k_i\leq n$-1. Furthermore, this segregation is ensured by i) $\sum_{i\in W_p}\sum_{k_i=1}^{n-1} z_{ijk_i} = 1$, for each $j\in N'$ and ii) $\sum_{i\in\{M-W_p\}}\sum_{k_i=1}^{n-1} z_{ijk_i} = 1$ for each $j\in\{N-N'\}$. Finally, the partitioning cardinality $k(i)$ for each $i\in W_p$ is identified in the following way. If for an $i\in W_p$, $0 < y_{ik_i} < 1$ for exactly one k_i, then either $k(i) = k_i$ or $k(i)-1 = k_i$. If $0 < y_{ik_i} < 1$ for two or more k_i values, then the cardinalities associated with the fractional y values are evenly distributed between K_i^- and K_i^+.

The set $H_q\subseteq N'$ denotes the collection of jobs whose potential assignment to each $i\in W_p$, either with cardinalities, $k_i\in K_i^+$ or $k_i\in K_i^-$, are hidden. In addition, depending on the relative values of $|H_q|$ and p, assignments of some jobs $j\in\{N-H_q\}$ to $i\in W_p$ may also be hidden. Consequently, the coefficient $\alpha_{ijk_i} = 0$, in the p-ACM inequality for each such hidden assignment. The coefficients of all remaining z variables is equal to 1. The p-Agent Cardinality Matching (p-ACM) inequalities are broadly classified into two: a) *Complete* p-ACM, in which $|N'| = n' \geq 2^p$, and b) *Partial* p-ACM inequalities in which $n' \leq 2^p$-1. We first present the *Complete* p-ACM inequality given its simple structure. The *Partial* p-ACM inequalities consists of two different structures depending on the relative sizes of n', p and the partitioning cardinalities $k(i)$ for each $i \in W_p$.

3.1 Complete p-Agent Cardinality Matching Inequalities

In the *Complete* p-ACM, the r-h-s parameters of (14) are specified as:

i) $\beta_0 = p$-1, (37a)

ii) $\beta_{in} = n$-p+1 for each $i \in M$, (37b)

iii) For each $i \in W_p$, if n-p+1 $\leq k_i \leq n$-1, then $\beta_{ik_i} = n - p$, else $\beta_{ik_i} = k_i - 1$, (37c)

iv) For each $i \in \{M\text{-}W_p\}$, if $k_i \geq n$-p, then $\beta_{ik_i} = n - p$, else $\beta_{ik_i} = k_i$. (37d)

It is evident from (37a) and (37c) that whenever every agent in W_p is utilized, the r-h-s of *Complete* p-ACM will be n-1. If $k_i > n$-p jobs are assigned to an agent $i \in W_p$, while the remaining (n-k_i) jobs are assigned (n-k_i) agents in $\{W_p\text{-}i\}$, i.e., $k_{i'} = 1$ for each of the (n-k_i) agents, then as well the r-h-s is n-1. The same holds if $k_i \geq n$-p jobs are assigned to $i \in \{M\text{-}W_p\}$, while the remaining (n-k_i) jobs are assigned to (n-k_i) agents in W_p. In all other feasible instances, the r-h-s is at least n.

The central idea behind the *Complete* p-ACM inequality is that, when all agents in W_p are utilized, then its l-h-s is at most n-1. This is achieved by ensuring that the assignment of at least one $j \in H_q$ is hidden from every $i \in W_p$ and $i \in \{M\text{-}W_p\}$ in any feasible solution. Let $W_q(j) \subseteq W_p$ with $|W_q(j)| = q$, denote a unique selection of agents, that is associated with a $j \in H_q$ where $0 \leq q \leq p$. In *Complete* p-ACM, the potential assignment of a $j \in H_q$ is hidden from: i) each $i \in W_q(j)$, $k_i \in K_i^-$, ii) each $i \in \{W_p\text{-}W_q(j)\}$, $k_i \in K_i^+$, and iii) each $i \in \{M\text{-}W_p\}$, $1 \leq k_i \leq n$-1, provided a feasible solution is possible with such an assignment. Of course, the assignment of a $j \in H_q$ to an i with cardinality k_i being hidden implies that $\alpha_{ijk_i} = 0$. It follows that, $|H_q| = \sum_{q=0}^{p} pC_q = 2^p$. The nomenclature, '*Complete*', conveys the notion that the partitioning cardinalities $k(i)$ and $k(i)$-1 across all $i \in W_p$ are large enough to be able to assign all 2^p jobs in H_q to agents in W_p. To enable this, the *Complete* p-ACM inequality requires that $k_{min} = Min\{k(i)|i \in W_p\} \geq \lfloor (2^p - 1)/p \rfloor$+1. Therefore, $\sum_{i \in W_p} k(i) \geq 2^p$+$p$. In addition, $n' = \sum_{i \in W_p} k(i) - r$ for $1 \leq r \leq p$-1. This points to feasible solutions in which all jobs in n' are assigned to agents in W_p, wherein for each $i \in W_p$ the active cardinalities are, $k_i = k(i)$ or $k_i = k(i)$-1.

Consider the assignment of $j \in H_q$ to $i \in \{W_p\text{-}W_q(j)\}$ for a $k_i \in K_i^+$. For $q \leq p$–2, a feasible solution with such an assignment is possible if

$$\sum_{i' \in \{W_p - W_q(j) - i\}} k(i') + k_i + q \leq n. \tag{38}$$

Note that the l-h-s in (38) represents the sum of $k(i')$ jobs assigned to each $i' \in \{W_p\text{-}W_q(j)\text{-}i\}$, one job assigned to each agent in $W_q(j)$ amounting to q, and k_i jobs assigned to i. Clearly, (38) must be satisfied in any feasible solution in which k_i jobs are assigned to i with all agents in W_p utilized. Let $k_{imax1}(q)$ = n-q-$\sum_{i' \in \{W_p - W_q(j) - i\}} k(i')$. If $k_i \leq k_{imax1}(q)$, $\alpha_{ijk_i} = 0$, else $\alpha_{ijk_i} = 1$. In the case of $q = p$-1, (38) is automatically satisfied for each $k_i \leq n$-p+1. When $k_i > n$-p+1, the remaining n-k_i jobs are less than q. Here, feasibility is achieved by assigning each of the (n-k_i) jobs to (n-k_i) agents in $W_q(j)$. Thus, in this case, $\alpha_{ijk_i} =$ 0 for all $k_i \in K_i^+$. Now consider the assignment of a $j \in H_q$ to $i \in \{M\text{-}W_p\}$ for which $q \leq p$-1. Here, if

$$\sum_{i' \in \{W_p - W_q(j)\}} k(i') + k_i + q \leq n, \tag{39}$$

then $\alpha_{ijk_i} = 0$, else $\alpha_{ijk_i} = 1$. As with (38), satisfaction of (39) implies that a feasible solution is possible with assigning $j \in H_q$ to $i \in \{M\text{-}W_p\}$ for a given k_i. Thus, given that $k_{imax2}(q)$ = n-q-$\sum_{i' \in \{W_p - W_q(j)\}} k(i')$, if $k_i \leq k_{imax2}(q)$, $\alpha_{ijk_i} = 0$, else $\alpha_{ijk_i} = 1$. The hidden assignments for each $j \in H_q$ are as follows:

a) assignment of $j \in H_q$ to each $i \in W_q(j)$ with $k_i \leq k(i)$-1, (40)

b) assignment of $j \in H_q$ to each $i \in \{W_p\text{-}W_q(j)\}$ for $k(i) \leq k_i \leq k_{imax1}(q)$, (41)

c) assignment of $j \in H_q$ to each $i \in \{M\text{-}W_p\}$, with $1 \leq k_i \leq k_{imax2}(q)$ if $q \leq p$-1, else with $1 \leq k_i \leq n$-1. (42)

For notational convenience, let $H_q^-(l)$ = $\{j \in H_q | \; q = l\}$. The *Complete p*-ACM inequality can be stated as

$$\begin{aligned}
&\sum_{i \in M} \sum_{j \in \{N - H_q\}} \sum_{k_i = 1}^{n-1} z_{ijk_i} + \sum_{i \in W_q(j)} \sum_{j \in H_q} \sum_{k_i \in K_i^+} z_{ijk_i} + \sum_{i \in \{W_p - W_q(j)\}} \sum_{j \in H_q} \sum_{k_i \in K_i^-} z_{ijk_i} \\
&\quad + \sum_{i \in \{W_p - W_q(j)\}} \sum_{q=0}^{p-2} \sum_{j \in H_q^-(q)} \sum_{k_i = k_{max1}(q)+1}^{n-1} z_{ijk_i} + \sum_{i \in \{M - W_p\}} \sum_{q=0}^{p-1} \sum_{j \in H_q^-(q)} \sum_{k_i = k_{max2}(q)+1}^{n-1} z_{ijk_i} \\
&\quad + \sum_{i \in M} \sum_{j \in N} z_{ijn} \\
&\quad \leq \sum_{i \in W_p} \sum_{k_i = 1}^{n-p} (k_i - 1) y_{ik_i} + \sum_{i \in W_p} \sum_{k_i = n-p+1}^{n-1} (n-p) y_{ik_i} + \sum_{i \in \{M - W_p\}} \sum_{k_i = 1}^{n-p-1} k_i y_{ik_i} \\
&\quad + \sum_{i \in \{M - W_p\}} \sum_{k_i = n-p}^{n-1} (n-p) y_{ik_i} + \sum_{i \in M} (n-p+1) y_{in} + (p-1)
\end{aligned} \tag{43}$$

The following example illustrates the construction of the *Complete p*-ACM inequality.

Example 3. Consider a (**P_{zy}**) with m = 4, n = 12 and W_p = {1, 2, 3}. Since $|W_p| = p = 3$, $|H_q| = 2^3 = 8$. Let H_q = {1, 2, …., 8} and $k(1) = k(2) = k(3) = 4$. Here, $W_0(1) = \phi$, $W_1(2)$ = {3}, $W_1(3)$ = {2}, $W_1(4)$ = {1}, $W_2(5)$ = {2, 3}, $W_2(6)$ = {1, 3}, $W_2(7)$ = {1, 2} and $W_3(8)$ = {1, 2, 3}. Here, $\beta_0 = 2$, $\alpha_{ij12} = 1$ for each $i \in M$, $j \in N$ and $\beta_{i12} = 10$ for each $i \in M$. The remaining β values are: i) for i = 1, 2 and 3, $\beta_{ik_i} = k_i - 1$, k_i = 1,.., 10, and $\beta_{ik_i} = 9$, k_i = 10, 11, ii) for i = 4, $\beta_{4k_4} = k_4$, for k_4 = 1,.., 8, and $\beta_{4k_4} = 9$, k_4 = 9, 10 and 11.

The following constitute the set of hidden assignments: i) For j = 1, its assignment to each $i\in\{1, 2, 3\}$ with k_i = 4 are hidden. That is because $W_0(1) = \phi$, and therefore as per (38), $k_{imax1}(0)$ = 4 for i = 1, 2 and 3. ii) For j = 2, as per (40), its assignment to i = 3 with $1 \le k_3 \le 3$ are hidden. Furthermore, $k_{imax1}(1)$ = 7 and $k_{imax2}(1)$ = 3. Therefore, its assignment to $i\in\{1, 2\}$ are hidden for $4 \le k_i \le 7$, while its assignment to i = 4 is hidden for $1 \le k_4 \le 3$. Similarly, for j = 3 its assignment to i = 2 is hidden for $1 \le k_2 \le 3$, its assignment to $i\in\{1, 3\}$ are hidden for $4 \le k_i \le 7$, while its assignment to i = 4 is hidden for $1 \le k_4 \le 3$. For j = 4 its assignment to i = 1 with $1 \le k_1 \le 3$, to $i\in\{2, 3\}$ with cardinalities $4 \le k_i \le 7$, and to i = 4 with $1 \le k_4 \le 3$, are all hidden. iii) For j = 5, as per (40), its assignment to $i\in\{2, 3\}$ with $1 \le k_i \le 3$, are hidden. For j = 5, $k_{imax2}(2)$ = 6. Its assignment to i = 1 with $4 \le k_1 \le 11$, and to i = 4 with $1 \le k_4 \le 6$ are hidden. Similarly, for j = 6, its assignment to $i\in\{1, 3\}$ with $1 \le k_i \le 3$, to i = 2 with $4 \le k_2 \le 11$, and to i = 4 with $1 \le k_4 \le 6$ are hidden. For j = 7, its assignment to $i\in\{1, 2\}$ with $1 \le k_i \le 3$, to i = 3 with $4 \le k_3 \le 11$, and to i = 4 with $1 \le k_4 \le 6$ are hidden. For j = 8, its assignment to $i\in\{1, 2, 3\}$ with $1 \le k_i \le 3$, and to i = 4 with $1 \le k_4 \le 11$, are hidden. □

Proposition 3.1 Every $(z, y)\in H(z, y)$ satisfies the *Complete p*-Agent Cardinality Matching inequality (43).

Proof: To begin with, it is worth noting that in any given $(z, y)\in H(z, y)$, the corresponding l-h-s of (43) is at most n. Clearly, the m solutions: y_{in} = 1, z_{ijn} = 1 for each $j\in N$, satisfy (43) as an equality. All remaining feasible solutions in ($\mathbf{P}_{zy}$) consist of active agents $|M'(l)| \ge 2$.

I) Consider feasible solutions in which, i) $|M'(l)\cap W_p| < p$, ii) $k_i < n\text{-}p+1$ for each $i\in\{M'(l)\cap W_p\}$ and iii) $k_i < n\text{-}p$ for each $i\in M'(l)\cap\{M\text{-}W_p\}$. Therefore, the r-h-s of (43), $\sum_{i\in\{M'(l)\cap W_p\}}(k_i - 1)+\sum_{i\in\{M'(l)\cap\{M-W_p\}\}} k_i+(p\text{-}1) \ge n$. Hence, for such solutions, (43) is satisfied.

II) Consider feasible solutions in which for some $i\in\{M'(l)\cap W_p\}$, $k_i \ge n\text{-}p+1$. For such solutions, the r-h-s of (43) will be n-1 only when the remaining $n\text{-}k_i$ jobs are assigned to $n\text{-}k_i$ agents in $\{W_p\text{-}i\}$. There exists a $j\in H_q$ with $W_q(j) = \{W_p\text{-}i\}$. Its assignment to i with $k_i \ge k(i)$ and to each $i'\in\{W_p\text{-}i\}$ with $k_{i'} \le k(i')\text{-}1$ are hidden. This results in the l-h-s of (43) being at most n-1, thereby satisfying it.

III) Consider feasible solutions in which for a $i\in M'(l) \cap\{M\text{-}W_p\}$, $k_i \ge n\text{-}p$. Here, the r-h-s of (43) will be n-1, only if the remaining $n\text{-}k_i$ jobs are assigned to $n\text{-}k_i$ agents in W_p. Since $n\text{-}k_i \le p$, such an assignment is possible. The assignment of $j\in H_q$ for which $W_q(j) = W_p$, is hidden from $i\in\{M\text{-}W_p\}$ with $k_i \ge n\text{-}p$ and from each $i'\in W_p$ with $k_{i'} \le k(i')\text{-}1$. Therefore, here as well the l-h-s of (43) is at most n-1, resulting in (43) being satisfied.

IV) Finally, consider feasible solutions in which $|M'(l)\cap W_p| = p$. The r-h-s of (43) corresponding to such solutions is n-1. In each such solution, there exists a $V_p\subseteq W_p$ with $|V_p| = q$, in which, i) for each $i\in V_p$, $k_i \le k(i)\text{-}1$, ii) for each $i\in\{W_p\text{-}V_p\}$ $k(i) \le k_i \le k_{imax1}(q)$, and iii) for each $i\in\{M'(l)\text{-}W_p\}$ $1 \le k_i \le k_{imax2}(q)$. However, associated with each such solution, there exists a $j\in H_q$ whose assignment to agents, $i\in V_p$, $k_i \le k(i)\text{-}1$, $i\in\{W_p\text{-}V_p\}$ $k(i) \le k_i \le k_{imax1}(q)$, and $i\in\{M'(l)\text{-}W_p\}$, $1 \le k_i \le k_{imax2}(q)$ are hidden. Thus, corresponding to these solutions as well, the l-h-s of (43) is at most n-1. □

Theorem 3.1 The *Complete p*-Agent Cardinality matching inequality (43) is a non-trivial facet of $H(z, y)$.

Proof: As per Proposition 2.2, non-trivial facets of $H(z, y)$ are constructed around $\widehat{M} \subseteq M$, with $|\widehat{M}| \geq 2$. In (43), $W_p \subseteq M$, with $p \geq 2$, and therefore non-trivial facets of $H(z, y)$ can be defined by setting $\widehat{M} = W_p$. From our discussion in Section 2.2, in any non-trivial facet of $H(z, y)$, α_{ijk_i}=0 for each $j \in N_{i-k_i}^{min}$, while α_{ijk_i}=1 for each $j \in \{N\text{-}N_{i-k_i}^{min}\}$, $i \in M$, $1 \leq k_i \leq n$-1. In addition, due to Lemma 2.6, $\beta_{ik_i} = \beta_{k_i}$ for all $i \in \{M\text{-}W_p\}$. As well, since $\alpha_{\hat{\imath}jk_{\hat{\imath}}}$=0 for each $j \in N_{\hat{\imath}-k_{\hat{\imath}}}^{min}$, due to minimality of non-trivial facets, $\beta_{\hat{\imath}k_{\hat{\imath}}} = \hat{\beta}_{k_{\hat{\imath}}}$ for all $\hat{\imath} \in W_p$, for each $1 \leq k_{\hat{\imath}} \leq n$-1. Thus, all minial, non-trivial facets of $H(z, y)$ take the form:

$$\sum_{\hat{\imath} \in W_p} \sum_{j \in \{N - N_{\hat{\imath}-k_{\hat{\imath}}}^{min}\}} \sum_{k_{\hat{\imath}}=1}^{n-1} z_{\hat{\imath}jk_{\hat{\imath}}} + \sum_{i \in \{M-W_p\}} \sum_{j \in \{N - N_{i-k_i}^{min}\}} \sum_{k_i=1}^{n-1} z_{ijk_i} + \sum_{i \in M} \sum_{j \in N} z_{ijn}$$
$$\leq \sum_{\hat{\imath} \in W_p} \sum_{k_{\hat{\imath}}=1}^{n-1} \hat{\beta}_{k_{\hat{\imath}}} y_{\hat{\imath}k_{\hat{\imath}}} + \sum_{i \in \{M-W_p\}} \sum_{k_i=1}^{n-1} \beta_{k_i} y_{ik_i} + \beta_0. \tag{44}$$

In (43), for each $\hat{\imath} \in W_p$, if $k_{\hat{\imath}} \in K_{\hat{\imath}}^-$ then $N_{\hat{\imath}-k_{\hat{\imath}}}^{min} = \{j \in H_q |\ \hat{\imath} \in W_q(j)\}$, while if $k_{\hat{\imath}} \in K_{\hat{\imath}}^+$ then $N_{\hat{\imath}-k_{\hat{\imath}}}^{min} = \{j \in H_q |\ \hat{\imath} \in \{W_p\text{-}W_q(j)\}$ and $k(\hat{\imath}) \leq k_{\hat{\imath}} \leq k_{imax1}(q)\}$. For each $i \in \{M\text{-}W_p\}$, $N_{i-k_i}^{min} = \{j \in H_q |\ 1 \leq k_i \leq k_{imax2}(q)\}$. Thus, as with (44), neither $N_{\hat{\imath}-k_{\hat{\imath}}}^{min}$ nor $N_{i-k_i}^{min}$ in (43) are null.

Assume that (43) is not a facet. Therefore, there exists a non-trivial facet (44), comprising of sets, W_p, $N_{\hat{\imath}-k_{\hat{\imath}}}^{min}$ for $\hat{\imath} \in W_p$, $1 \leq k_{\hat{\imath}} \leq n$-1, and $N_{i-k_i}^{min}$ for $i \in \{M\text{-}W_p\}$, $1 \leq k_i \leq n$-1, that are identical to that in (43). Let $F(z, y) = \{(z, y) \in H(z, y) |$ (44) is satisfied as an equality$\}$, while $G(z, y) = \{(z, y) \in H(z, y) |$ (43) is satisfied as an equality$\}$. Since (44) is a facet while (43) is not, $G(z, y) \subset F(z, y)$. Thus, if $(z, y) \in G(z, y)$, then $(z, y) \in F(z, y)$ as well. In the arguments to follow, we examine a set of integer solutions $(z, y) \in G(z, y)$, using which (44) is shown to be a scalar multiple of (43). That being so, the only way (43) is not a facet is if one or more sets $N_{\hat{\imath}-k_{\hat{\imath}}}^{min}$ and $N_{i-k_i}^{min}$ in (44) are strict subsets of corresponding sets in (43). Using sequential lifting, coefficients $\alpha_{\hat{\imath}jk_{\hat{\imath}}}$ for each $\hat{\imath} \in W_p$, $1 \leq k_{\hat{\imath}} \leq n$-1, $j \in N_{\hat{\imath}-k_{\hat{\imath}}}^{min}$, as well as α_{ijk_i} for each $i \in \{M\text{-}W_p\}$, $1 \leq k_i \leq n$-1, $j \in N_{i-k_i}^{min}$, are attempted to be lifted. If after lifting, all these coefficients remain zero, then (43) is indeed a non-trivial facet of $H(z, y)$.

The following are the set of integer solutions $(z, y) \in H(z, y)$, referred to above:

a) $(z, y)^1$: $y_{ik_i} = 1$, for some $i \in \{M\text{-}W_p\}$, $n\text{-}p \leq k_i \leq n$-1, $y_{\hat{\imath}1} = 1$ for each $\hat{\imath} \in M_p$, $M_p \subseteq W_p$ and $|M_p| = n\text{-}k_i$.

b) $(z, y)^2$: $y_{\hat{\imath}_1 k_{\hat{\imath}_1}} = 1$, $\hat{\imath}_1 \in W_p$, $n\text{-}p+1 \leq k_{\hat{\imath}_1} \leq n$-1, $y_{\hat{\imath}1} = 1$ for each $\hat{\imath} \in M_p$, $M_p \subseteq \{W_p\text{-}\hat{\imath}_1\}$ and $|M_p| = n\text{-}k_{\hat{\imath}_1}$.

d) $(z, y)^3$: For some $0 \leq q \leq p$-1 with a selection of $W_q(j^*) \subset W_p$ in which $|W_q(j^*)| = q$, $y_{\hat{\imath}k(\hat{\imath})-1}$= 1 for each $\hat{\imath} \in W_q(j^*)$, $y_{\hat{\imath}k(\hat{\imath})} = 1$ for each $\hat{\imath} \in \{W_p\text{-}W_q(j^*)\}$, and $y_{ik_i} = 1$ for some $i \in \{M\text{-}W_p\}$, where $k_i = n - \sum_{\hat{\imath} \in \{W_p - W_q(j^*)\}} k(\hat{\imath}) - \sum_{\hat{\imath} \in W_q(j^*)} (k(\hat{\imath}) - 1)$.

We first show that $(z, y)^s \in G(z, y)$, $s = 1,.., 3$. It is readily apparent that the r-h-s of (43) obtained by substituting each $(z, y)^s$, $s = 1,.., 3$ in it is $(n$-1). In the case of $(z, y)^1$, let $j^* \in H_q$ denote the job for which $|W_q(j^*)| = q = p$, and whose assignment to each $\hat{\imath} \in W_p$ with cardinality $k_{\hat{\imath}} \in K_{\hat{\imath}}^-$, and to each $i \in \{M\text{-}W_p\}$ with $n\text{-}p \leq k_i \leq n$-1 is hidden. In $(z, y)^1$, j^* and all the remaining jobs $j \in \{H_q\text{-}j^*\}$ are assigned to $i \in \{M\text{-}W_p\}$ with

n-$p \le k_i \le n$-1, while all $j \in \{N$-$H_q\}$ are assigned to the remaining slots in i-k_i and to each $\hat{\imath} \in W_p$. Since $n \ge |H_q|+p$, such an assignment is possible. For each $j \in \{H_q$-$j^*\}$, $|W_q(j)| \le p$-1 and therefore its associated $k_{imax2}(q) \le n$-p-1. This implies that the assignment of no job $j \in \{H_q$-$j^*\}$ is hidden from any $i \in \{M$-$W_p\}$ with n-$p \le k_i \le n$-1. On the other hand, the assignment of j^* is hidden from precisely those agent-cardinality sets that are active in $(z, y)^1$. Thus the l-h-s of (43) is exactly n-1. In $(z, y)^2$ there exists a $j^* \in H_q$ whose $|W_q(j^*)| = p$-1, and whose assignment to $\hat{\imath}_1 \in W_p$, n-p+1 $\le k_{\hat{\imath}_1} \le n$-1, and each $\hat{\imath} \in \{W_p$-$\hat{\imath}_1\}$ with $k_{\hat{\imath}} \in K_{\hat{\imath}}^-$ is hidden. In $(z, y)^2$, all $j \in H_q$ including j^* is is assigned to $\hat{\imath}_1$. Since $n \ge |H_q|+p$, such an assignment is possible. Clearly, all $j \in \{H_q$-$j^*\}$ for which $\hat{\imath}_1 \in W_q(j)$, its assignment to $\hat{\imath}_1$ with $k_{\hat{\imath}_1} \ge n$-p+1 is not hidden. Next, all jobs $j \in \{H_q$-$j^*\}$ for which $\hat{\imath}_1 \notin W_q(j)$, its $|W_q(j)| = q \le p$-2. For such jobs, it follows from (38) that $k_{imax1}(q) \le n$-p and therefore their assignment to $\hat{\imath}_1$ with $k_{\hat{\imath}_1} \ge n$-p+1 is not hidden. The remaining jobs $\{N$-$H_q\}$ along with j^* are assigned to the remaining slots in $\hat{\imath}_1$ and to each $\hat{\imath} \in M_p$, $M_p = \{W_p$-$\hat{\imath}_1\}$. Since only the assignment of j^* is hidden in $(z, y)^2$, the l-h-s of (43) is exactly n-1.

In $(z, y)^3$, the $j \in H_q$, whose $|W_q(j)| = p$, be assigned to an $\hat{\imath} \in \{W_p$-$W_q(j^*)\}$. Each $j \in H_q$ for which $|W_q(j)| = p$-1 is assigned to a $\hat{\imath} \notin W_q(j)$, all with cardinalities of $k_{\hat{\imath}} \in K_{\hat{\imath}}^-$. That $j \in H_q$ for which $q = 0$, is assigned to a $\hat{\imath} \in W_q(j^*)$ with a cardinality of $k(\hat{\imath})$-1. None of these assignments are hidden. The remaining jobs in H_q with $1 \le q \le p$-2, are assigned as follows. For a $j \in \{H_q$-$j^*\}$, if there exists a $\hat{\imath} \in W_q(j)$ such that $\hat{\imath} \notin W_q(j^*)$, then j is assigned to $\hat{\imath}$ with cardinality $k(\hat{\imath})$. If such an agent does not exist, then there must a $\hat{\imath} \notin W_q(j)$ such that $\hat{\imath} \in W_q(j^*)$, in which case j is assigned to $\hat{\imath}$ with $k_{\hat{\imath}} = k(\hat{\imath})$-1. This ensures that none of these assignments are hidden. Furthermore, since $\sum_{i \in W_p} k(i) \ge |H_q|+p$, such an assignment is definitely possible. Finally, all jobs in $\{N$-$H_q\}$, along with j^* are assigned to remaining slots in W_p and $i \in \{M$-$W_p\}$. Therefore, associated with $(z, y)^3$, the l-h-s of (43) is exactly equal to n-1. Thus, $(z, y)^s$ for $s = 1, \ldots, 3$ all belong to $G(z, y)$.

Consider an instance of $(z, y)^1 \in G(z, y)$, in which $k_i = n$-1, $|M_p| = 1$ and $\hat{\imath}_1 \in M_p$. Substituting this solution in (44), one obtains its r-h-s to be, $\beta_{\hat{\imath}_1 1}$+$\beta_{i,n-1}$+β_0. Due to Corollary 2.2 and Lemma 2.6, $\beta_{\hat{\imath}1}$ will be referred to as $\hat{\beta}_1$, for each $\hat{\imath} \in W_p$ and β_{ik_i} as β_{k_i} for each $i \in \{M$-$W_p\}$, respectively. Since $(z, y)^1 \in G(z, y)$, the l-h-s of (44) is n-1, where $j^* \in H_q$ corresponds to $|W_q(j^*)| = p$. This solution is perturbed by introducing another $\hat{\imath} \in \{W_p$-$M_p\}$ with a cardinality of 1 while reducing the cardinality k_i to n-2. The perturbed solution also belongs to $G(z, y)$. Consequently, with the perturbed solution, the l-h-s remains n-1, and the r-h-s of (44) becomes $2\hat{\beta}_1$+β_{n-2}+β_0. Therefore, $\hat{\beta}_1$+β_{n-1}+β_0 = $2\hat{\beta}_1$+β_{n-2}+β_0, resulting in β_{n-2}=β_{n-1}-$\hat{\beta}_1$. By sequentially introducing a $\hat{\imath} \in \{W_p$-$M_p\}$ and reducing the cardinality of k_i by 1, one obtains

$$\beta_{n-l} = \beta_{n-l+1} - \hat{\beta}_1, l = 2, \ldots, p. \tag{45}$$

Perturbing $(z, y)^2$ in a similar way, one obtains the result

$$\beta_{\hat{\imath},n-l} = \beta_{\hat{\imath},n-l+1} - \hat{\beta}_1, l = 2, \ldots, p-1. \tag{46}$$

Since the l-h-s of (44) remains n-1 both for $(z, y)^1$ and $(z, y)^2$, $\beta_{\hat{\imath},n-l} = \hat{\beta}_{n-l} = \beta_{n-l}$, for $l = 2, \ldots, p$-1.

Now consider an instance of $(z, y)^1 \in G(z, y)$ in which $k_i = n\text{-}p$ and $M_p = W_p$. This solution is perturbed by introducing $i' \in \{M\text{-}W_q\text{-}i\}$ with $k_{i'} = 1$, while reducing the cardinality $k_i = n\text{-}p\text{-}1$. The job that was assigned to i in $(z, y)^1$ is now assigned to i'. The perturbed solution belongs to $G(z, y)$ as well with the same $j^* \in H_q$ being hidden from all active agents in both solutions. Therefore, with both solutions, the l-h-s of (44) remains n-1, and $\beta_{n-p-1} = \beta_{n-p}\text{-}\beta_1$. This result can be generalized by progressively transferring jobs assigned to i, to a new agent $i' \in \{M\text{-}W_q\}$. Thus, $\beta_{k_i-1} = \beta_{k_i}\text{-}\beta_1$, for each $p \le k_i \le n\text{-}p\text{-}1$. Now consider a variant of $(z, y)^1 \in G(z, y)$ in which $q = p$ with $y_{\hat{\imath}k(\hat{\imath})-1} = 1$ for each $\hat{\imath} \in W_p$. Here, for some $i \in \{M\text{-}W_p\}$, $k_i = n\text{-}\sum_{\hat{\imath}\in W_p}(k(\hat{\imath}) - 1) \ge p$. Here as well, by introducing a $i' \in \{M\text{-}W_q\text{-}i\}$ with $k_{i'} = 1$, and reducing the cardinality k_i by 1, we obtain the result $\beta_{k_i} = \beta_{k_i+1} - \beta_1$, for $1 \le k_i \le n\text{-}\sum_{\hat{\imath}\in W_p}(k(\hat{\imath}) - 1)$. Thus, in addition to (45), the general result is

$$\beta_{k_i} = \beta_{k_i+1} - \beta_1, k_i = 1, \dots, n - p - 1. \tag{47}$$

Now consider a $(z, y)^3 \in G(z, y)$ in which $k_{\hat{\imath}_1} = n\text{-}p\text{+}1$ and $M_p = \{W_p\text{-}\hat{\imath}_1\}$ with $y_{\hat{\imath}1} = 1$ for each $\hat{\imath} \in M_p$. This solution is perturbed by reassigning a job j that was assigned to $\hat{\imath}_1$ to an $i \in \{M\text{-}W_p\}$, to obtain another $(z, y)^3 \in G(z, y)$. In this solution, $k_{\hat{\imath}_1} = n\text{-}p$ and $k_i = 1$. In both solutions, the same $j^* \in H_q$ is hidden from all active agents and therefore the l-h-s values in (44) remain unchanged. Consequently, $\hat{\beta}_{n-p+1} = \beta_{\hat{\imath}_1 n-p} + \beta_1$. Note that this relationship holds for every $\hat{\imath}_1 \in W_p$. Hence, $\hat{\beta}_{n-p+1} = \hat{\beta}_{n-p} + \beta_1$. This process of perturbation can be generalized, wherein there exists a $(z, y)^3 \in G(z, y)$ in which $2 \le k_{\hat{\imath}_1} \le n\text{-}p$. This solution is perturbed by reassigning jobs that was assigned to $\hat{\imath}_1$ to an unassigned agent $i' \in \{M\text{-}W_p\}$. Consequently, in the perturbed solution, $k_{\hat{\imath}_1} \to k_{\hat{\imath}_1}\text{-}1$, and $k_{i'} = 1$. Such perturbations provide us with the result,

$$\hat{\beta}_{k_{\hat{\imath}_1}} = \hat{\beta}_{k_{\hat{\imath}_1}+1} - \beta_1, \text{ for } 1 \le k_{\hat{\imath}_1} \le n\text{-}p. \tag{48}$$

Based on (47) and (48), it is clear that regardless of whether $(z, y)^1$, $(z, y)^2$ or $(z, y)^3$ is substituted in (44), its r-h-s is

$$p\hat{\beta}_1 + (n - p)\beta_1 + \beta_0 = n - 1. \tag{49}$$

If in $(z, y)^1$, $\hat{\imath}$ is replaced by $i' \in \{M\text{-}W_p\text{-}i\}$, with $y_{i'1} = 1$, the resulting solution satisfies (44) as a strict inequality. This suggests that $\beta_1 > \hat{\beta}_1$. That being so, $\hat{\beta}_1 = \beta_1\text{-}\Delta_1$, with $\Delta_1 > 0$. Therefore,

$$n\beta_1 - p\Delta_1 + \beta_0 = n - 1. \tag{50}$$

Clearly, $\beta_1 \ge \Delta_1 > 0$ and $\beta_0 > 0$ and integer. Since $n > p$ and $\beta_0 > 0$, $\beta_1 = 1$. Otherwise, (50) cannot be satisfied. Therefore, $\Delta_1 = 1$ and $\beta_0 = p\text{-}1$. Thus, (44) is identical to (43), provided one or more sets, $N_{i-k_i}^{min}$ and $N_{\hat{\imath}-k_{\hat{\imath}}}^{min}$ in (44) are not a subset of equivalent sets in (43). If in fact that is the case, sequential lifting is performed to lift each of the coefficients of variables that correspond to assignments that are hidden in (43). Let $\alpha' z \le \beta' y + (p\text{-}1)$ denote a compact representation of (43), and $WHK = \{i\text{-}j\text{-}k_i \mid i \in M, j \in N, 1 \le k_i \le n\text{-}1, \alpha_{ijk_i} = 0$ in (43)$\}$. To lift the coefficient of $z_{i'j'k_{i'}}$ whose $(i'\text{-}j'\text{-}k_{i'}) \in WHK$, the optimization problem solved is:

$$\alpha_{i'j'k_{i'}} = Min\ \{\beta' y + (p\text{-}1) - \alpha' z \mid (4)\text{-}(9), z_{i'j'k_{i'}} = 1\}. \tag{51}$$

Note that since (43) is a valid inequality, $\beta' y+(p-1)-\alpha' z \geq 0$ for every $(z, y) \in H(z, y)$. Furthermore, solutions $(z, y)^s$, $s = 1,.., 3$ satisfy (43) as an equality. Hence, regardless of the order in which indices i'-j'-$k_{i'}$ in WHK are considered, $\alpha_{i' j' k_{i'}} = 0$, for all $(i'$-j'-$k_{i'}) \in WHK$. Thus, (43) is a non-trivial facet of $H(z, y)$. □

We now shift our attention to identifying fractional solutions of $LP(z, y)$ (defined in (11)), that violate inequality (43). In fact, the fractional solutions that we identify violate (43) by the maximum possible amount of 1. Fractional solutions $(z, y)^f \in LP(z, y)$ are defined by $M' \subseteq M$ and $N' \subseteq N$ as follows:

i) $0 < y_{i\hat{k}_i-1} = \hat{y}_{K_L} < 1$, $0 < y_{i\hat{k}_i} = \hat{y}_{K_U} < 1$ and $\hat{y}_{K_L} + \hat{y}_{K_U} = 1$, for each $i \in M'$, (52a)

ii) $z_{ij\hat{k}_i-1} < 1$ and $z_{ij\hat{k}_i} < 1$ for each $i \in M'$, $j \in N'$, (52b)

iii) $\sum_{i\in M'} \sum_{k_i=1}^{n-1} z_{ijk_i} = 1$ for each $j \in N'$, (52c)

iv) $\sum_{i\in M'} \sum_{k_i=1}^{n-1} z_{ijk_i} = 0$ for each $j \in \{N\text{-}N'\}$. (52d)

The conditions in (52a) and (52b) highlight the fractional y values in $(z, y)^f$, and that all agents $i \in M'$ are fully utilized. Condition (52c) suggests that each $j \in N'$ is 'assigned' entirely to agents in M' albeit fractionally, while (52d) suggests that each job in $\{N\text{-}N'\}$ is 'assigned' entirely to agents in $\{M\text{-}M'\}$. For $(z, y)^f$ to violate (43), we set $W_p \to M'$ and $k(i) \to \hat{k}_i$ for each $i \in M'$. From (52c) and (52d), we obtain,

$$\sum_{i\in M'} ((\hat{k}_i - 1)\hat{y}_{K_L} + \hat{k}_i \hat{y}_{K_U}) = n'. \tag{53}$$

Since $\hat{y}_{K_L} + \hat{y}_{K_U} = 1$ and $m' = p$, (53) reduces to $\sum_{i\in M'} \hat{k}_i = n' + p\hat{y}_{K_L}$. Both n' and $\hat{k}_i$ for each $i \in M'$ being positive integers, it follows that $r = p\hat{y}_{K_L}$ is a positive integer as well. With $0 < \hat{y}_{K_L} < 1$, p-1 distinct fractional solutions are possible, one for each value of $1 \leq r \leq p$-1. Thus, for each $1 \leq r \leq p$-1, $y_{i\hat{k}_i-1} = \hat{y}_{K_L} = r/p$ and $y_{i\hat{k}_i} = \hat{y}_{K_U} = (p-r)/p$ for each $i \in M'$. Consequently, the r-h-s of (43) is n-1.

The z values in $(z, y)^f$ can be viewed as flows in a directed bipartitite network consisting of nodes in $[M'_{\hat{k}-1}$, $M'_{\hat{k}}$, $N']$ and arcs in $A_{m'n'}$. Each arc $(r, s) \in A_{m'n'}$ has $r \in \{M'_{\hat{k}-1}, M'_{\hat{k}}\}$ and $s \in N'$. Nodes in $M'_{\hat{k}-1}$ and $M'_{\hat{k}}$ represent agent-cardinality combinations defined as, $M'_{\hat{k}-1} = \{i\text{-}(\hat{k}_i\text{-}1) | \ i \in M'\}$ and $M'_{\hat{k}} = \{i\text{-}\hat{k}_i | \ i \in M'\}$, respectively. Thus, $|M'_{\hat{k}-1}| = |M'_{\hat{k}}| = p$. Each i-$(\hat{k}_i-1) \in M'_{\hat{k}-1}$ and i-$\hat{k}_i \in M'_{\hat{k}}$ provides a 'supply' of $(\hat{k}_i - 1)r/p$ and $\hat{k}_i(p-r)/p$ units, respectively. Each $j \in N'$ has a 'demand' of 1. It follows from (53) that the total supply provided by nodes in $M'_{\hat{k}-1}$ and $M'_{\hat{k}}$ equals the total demand required of nodes in N'. One set of arcs in $A_{m'n'}$ that emanate from i-$(\hat{k}_i$-1) has a capacity of $\hat{y}_{K_L}$. The other set of arcs that emanate from i-$\hat{k}_i$ has a capacity of $\hat{y}_{K_U}$ each. Thus, each arc in $A_{m'n'}$ represents potential unhidden assignments of jobs in N' to agents in M' with a cardinality of either $(\hat{k}_i$-1) or $\hat{k}_i$. Note that the assignment of each $j \in H_q$ is hidden from $|W_q(j)| = q$ agent-cardinality combinations in $M'_{\hat{k}-1}$ and $|M'\text{-}W_q(j)| = p - q$ agent-cardinality combinations in $M'_{\hat{k}}$. Thus, q arcs from $M'_{\hat{k}}$ and $(p - q)$ arcs from $M'_{\hat{k}-1}$ enter each $j \in H_q$, amounting to p arcs, while $2p$ arcs enter each $j \in \{N'\text{-}H_q\}$. Recall that for a given $0 \leq q \leq p$, there are pC_q unique selections of $W_q(j) \subseteq M'$, each associated with a $j \in H_q$. Let $j_q(l) \in H_q$, where $l = 1,\ldots, pC_q$, denote a job whose assignment is hidden from each $i \in W_q(j_q(l))$ with $k_i \leq k(i)$-1 and each $i \in \{W_p\text{-}W_q(j_q(l))\}$ with $k_i \geq k(i)$. Therefore, the total number of arcs leaving each i-$\hat{k}_i \in M'_{\hat{k}}$ and entering $j_q(l) \in H_q$, for a given $1 \leq q \leq p$-1, is $p-1C_{q-1}$. The total number of arcs leaving each i-($\hat{k}_i$-1)∈$M'_{\hat{k}-1}$ and entering nodes $j_q(l)$∈H_q, is $p-1C_{p-q}$. For q = 0, one arc leaves each i-($\hat{k}_i$-1)∈$M'_{\hat{k}-1}$ and enters one node $j_0(1)$, and for q = p, one arc leaves each i-$\hat{k}_i$∈$M'_{\hat{k}}$ and enters $j_p(1)$.

Let, i) $R\_Cap_{\hat{k}_i-1}$ = $(\hat{k}_i-1)\hat{y}_{K_L}$ and $R\_Cap_{\hat{k}_i}$ = $\hat{k}_i\hat{y}_{K_U}$, represent the available capacity at each i-($\hat{k}_i$-1)∈$M'_{\hat{k}-1}$ and i-$\hat{k}_i$∈$M'_{\hat{k}}$, respectively, and ii) N''⊆N' denote the currently unassigned jobs in N'. The following algorithm, specifies the z variables representing flows between {$M'_{\hat{k}-1}$, $M'_{\hat{k}}$} and N'.

***Algorithm_z{M', N'}*:**

I. Set $N'' \rightarrow N'$.

II. For each i∈M', if $\hat{k}_i$>k_{min}, do the following: i) Select $N_s(i)$ = {j∈{N''-H_q}| |$N_s(i)$| = ($\hat{k}_i$-k_{min})}, ii) Set a) $z_{ij\hat{k}_i-1} \rightarrow \hat{y}_{K_L}$ for each j∈$N_s(i)$, b) $z_{ij\hat{k}_i} \rightarrow \hat{y}_{K_U}$ for each j∈$N_s(i)$, c) $R\_Cap_{\hat{k}_i-1}$→ $R\_Cap_{\hat{k}_i-1}$- |$N_s(i)$|$\hat{y}_{K_L}$, $R\_Cap_{\hat{k}_i}$→ $R\_Cap_{\hat{k}_i}$- |$N_s(i)$|$\hat{y}_{K_U}$, and c) $N'' \rightarrow N''$- $N_s(i)$.

III. For each j∈H_q, do the following: i) If |$W_q(j)$| = q = 0, then set a) $z_{ij_0^*(1)\hat{k}_i-1} \rightarrow 1/p$ for each i∈M', and b) $R\_Cap_{\hat{k}_i-1}$→ $R\_Cap_{\hat{k}_i-1}-1/p$ for each i-($\hat{k}_i$-1)∈$M'_{\hat{k}-1}$. ii) If |$W_q(j)$| = q = p, then set a) $z_{ij_p^*(1)\hat{k}_i} \rightarrow 1/p$ for each i∈M', b) $R\_Cap_{\hat{k}_i}$→ $R\_Cap_{\hat{k}_i}-1/p$ for each i-$\hat{k}_i$∈$M'_{\hat{k}}$. iii) If 1 ≤ |$W_q(j)$| = q ≤ p-1, then for each l = 1,…, pC_q, a) $z_{ij_q^*(l)\hat{k}_i-1} \rightarrow \hat{y}_{K_L}/(p-q)$, $R\_Cap_{\hat{k}_i-1}$→ $R\_Cap_{\hat{k}_i-1}$-$\hat{y}_{K_L}/(p-q)$ for each i∈{M'-$W_q(j_q^*(l))$}, b) $z_{ij_q^*(l)\hat{k}_i} \rightarrow \hat{y}_{K_U}/q$, $R\_Cap_{\hat{k}_i}$→ $R\_Cap_{\hat{k}_i}$-$\hat{y}_{K_U}/q$ for each i∈$W_q(j_q^*(l))$.

IV. Set $N'' \rightarrow N''$-H_q.

V. If $R\_Cap_{\hat{k}_i-1}$> 0, then, i) select N_s = {j∈N''| |N_s| = $p(R\_Cap_{\hat{k}_i-1})$}, ii) Set $z_{ij\hat{k}_i-1} \rightarrow 1/p$ for each j∈N_s, i-($\hat{k}_i$-1)∈$M'_{\hat{k}-1}$, and iii) $N'' \rightarrow$ {N''-N_s}.

VI. If $R\_Cap_{\hat{k}_i}$> 0, then, i) set N_s = N'', ii) Set $z_{ij\hat{k}_i} \rightarrow 1/p$ for each j∈N_s, i-$\hat{k}_i$∈$M'_{\hat{k}}$.

Note that by definition, k_{min} = Min{$k(i)$| i∈M'} ≥ $\lceil |H_q|/p \rceil$+1. Therefore, $\sum_{i\in M'}(\hat{k}_i-|N_s(i)|) = pk_{min} \geq$ |H_q|+p, implying that in step II of ***Algorithm_z{M', N'}***, it is possible to extract $N_s(i)$ from {N'-H_q} for each i∈M'. Observe as well that the z values in step III are set such that, i) the total flow into each j∈H_q is equal to 1, and ii) the flow on each arc satisfies its arc capacity. Finally, since k_{min} ≥ $\lceil |H_q|/p \rceil$+1, p($R\_Cap_{\hat{k}_i-1}$+$R\_Cap_{\hat{k}_i}$) = pk_{min}- r > |H_q|. Thus, there is sufficient capacity at $M'_{\hat{k}-1}$ and $M'_{\hat{k}}$ to accommodate all jobs in H_q. With this, the r-h-s of (43) takes on a value of n-1. Since the z values associated with each hidden assignment equal to zero, $(z, y)^f$ constructed above violates (43) by the maximum amount of 1. The following example illustrates the fractional part of $(z, y)^f$ described above.

Example 4. Consider first an instance of $(z, y)^f$ in which M' = {1, 2, 3}, N' = {1,….., 12} and H_q = {1,…, 8}. Jobs j∈H_q follow the ordering: $j_0(1)$ = 1, $j_1(1)$ = 2, $j_1(2)$ = 3, $j_1(3)$ = 4, $j_2(1)$ = 5, $j_2(2)$ = 6, $j_2(3)$ = 7, $j_3(1)$ = 8. Here, p = 3, with $\hat{k}_1$ = 5, $\hat{k}_2$ = 4 and $\hat{k}_3$ = 4. Since n' = 12 it follows that r = 1 and the y values in $(z, y)^f$ are, y_{15} = y_{24} = y_{34} = 2/3, y_{14} = y_{23} = y_{33} = 1/3. In this example, $\hat{k}_{min}$ = 4. In step II, z_{194} = 1/3 and z_{195} = 2/3. On completion of step II, 1, $R\_Cap_{\hat{k}_i-1}$ = $(\hat{k}_{min}-1)r/p$ = 1 for each i-($\hat{k}_i$-1)∈$M'_{\hat{k}-1}$, and $R\_Cap_{\hat{k}_i}$= $k_{min}(p-r)/p$ = 8/3 for each i-$\hat{k}_i$∈$M'_{\hat{k}}$. In step III, i) for q = 0, z_{114} = z_{213} = z_{313} = 1/3, ii) for q = 3, z_{185} = z_{284} = z_{384} = 1/3, iii) for q = 1, z_{124} = z_{223} = 1/6, z_{324} = 2/3, z_{134} = z_{333} = 1/6, z_{234} = 2/3, z_{243} = z_{343} = 1/6, z_{145} = 2/3, and iv) for q = 2, z_{155} = z_{253} = z_{353} = 1/3, z_{164} = z_{264} = z_{363} = 1/3, z_{174} =

$z_{273} = z_{374} = 1/3$. On completion of step III, $R\_Cap_{\hat{k}_i-1} = 0$ for each i-($\hat{k}_i$-1)$\in M'_{\hat{k}-1}$, and $R\_Cap_{\hat{k}_i} = 3/3$ for each i-$\hat{k}_i \in M'_{\hat{k}}$. In step VI, $z_{1j5} = z_{2j4} = z_{3j4} = 1/3$ for j = 10, 11 and 12. It is clear from the partial solution so constructed, that the l-h-s of (43) exceeds its r-h-s by 1, resulting in the maximum possible violation of the inequality. The fractional solution above is presented in Figure 4 below.

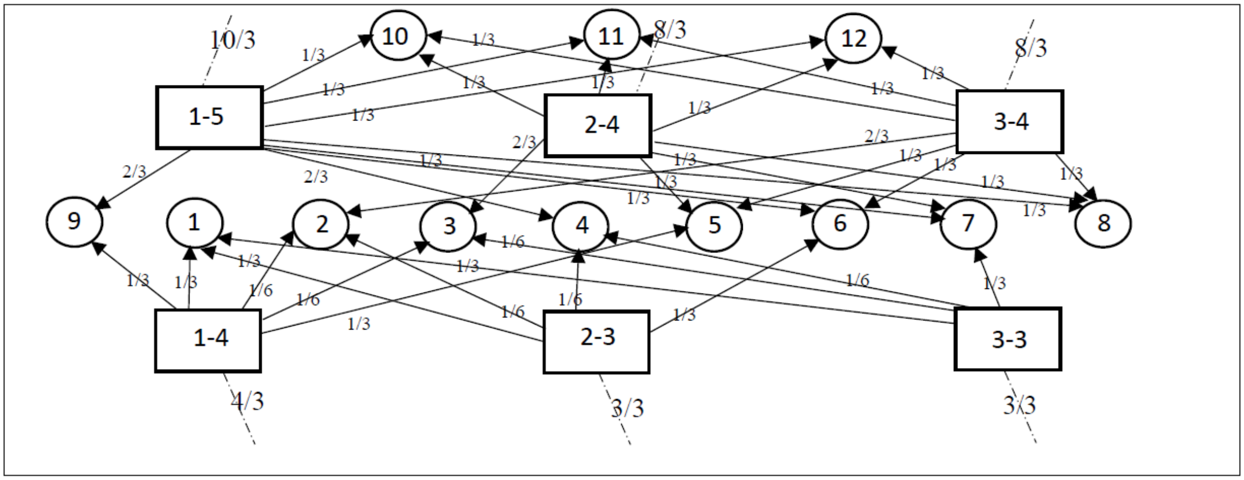

Figure 4. Illustration of a fractional solution that violates the *Complete* p-ACM inequality.

Consider another instance of $(z, y)^f$ in which $M' = \{1,\ldots, 4\}$, $N' = \{1,\ldots, 17\}$, $H_q = \{1,\ldots, 16\}$ and $\hat{k}_i = 5$ for $i = 1,\ldots, 4$. Naturally, $y_{15} = y_{25} = y_{35} = y_{45} = 1/4$, $y_{14} = y_{24} = y_{34} = y_{44} = 3/4$. Using the ***Algorithm_z{M', N'}***, the z values obtained are, $z_{114} = z_{214} = z_{314} = z_{414} = z_{1,16,5} = z_{2,16,5} = z_{3,16,5} = z_{4,16,5} = 1/4$, $z_{124} = z_{224} = z_{324} = z_{425} = z_{134} = z_{234} = z_{335} = z_{434} = z_{144} = z_{245} = z_{344} = z_{444} = z_{155} = z_{254} = z_{354} = z_{454} = 1/4$, $z_{164} = z_{264} = z_{174} = z_{374} = z_{184} = z_{484} = z_{294} = z_{394} = z_{2,10,4} = z_{4,10,4} = z_{3,11,4} = z_{4,11,4} = 3/8$, $z_{365} = z_{465} = z_{275} = z_{475} = z_{285} = z_{385} = z_{195} = z_{495} = z_{1,10,5} = z_{3,10,5} = z_{1,11,5} = z_{2,11,5} = 1/8$, $z_{1,12,4} = z_{2,13,4} = z_{3,14,4} = z_{4,15,4} = 3/4$, $z_{2,12,5} = z_{3,12,5} = z_{4,12,5} = z_{1,13,5} = z_{3,13,5} = z_{4,13,5} = z_{1,14,5} = z_{2,14,5} = z_{4,14,5} = z_{1,15,5} = z_{2,15,5} = z_{3,15,5} = 1/12$. Here as well, $(z, y)^f$ satisfies (4), (5), (7) and (8) by ensuring that no flow occurs on arcs associated with hidden assignments. Thus, (43) is violated by an amount of 1. □

3.2 Partial p-Agent Cardinality Matching Inequalities

As with the *Complete* p-ACM inequality, *Partial* p-ACM inequalities are also constructed around sets $W_p \subseteq M$ with $|W_p| = p$ and $H_q \subseteq N' \subseteq N$. As well, cardinalities of each $i \in W_p$ are partitioned into two sets: a) $K_i^- = \{1,\ldots, k(i)\text{-}1\}$, b) $K_i^+ = \{k(i),\ldots, n\text{-}1\}$. However, *Partial* p-ACM inequalities apply when $n' \le 2^p$-1. Further, their structure changes as n' decreases in relation to p. Broadly, *Partial* p-ACM inequalities can be classified into the following two cases: I) $p+1 \le n' \le \sum_{i\in W_p} k(i) - r$ for some $1 \le r \le p$-1 and II) $n' \le p$. Note that when $n' = \sum_{i\in W_p} k(i) - r$ for some $1 \le r \le p$-1, the *Partial* p-ACM inequality-Case I) seeks to violate fractional solutions $(z, y)^f$ in which $\sum_{k_i=1}^{n-1} y_{ik_i}=1$, for each $i \in W_p$, while if $n' \le \sum_{i\in W_p} k(i) - p$, $\sum_{k_i=1}^{n-1} y_{ik_i}<1$, for each $i \in W_p$. With *Partial* p-ACM inequality-Case II), in which $n' \le p$ as well, the fractional solution $(z, y)^f$, is such that $\sum_{k_i=1}^{n-1} y_{ik_i}<1$, for each $i \in W_p$. The two cases will be discussed in the same order.

Partial p-ACM inequality-*Case* I):

As with *Complete p*-ACM inequalities, conditions (37a), (37b), (37c) and (37d), apply here as well. As well, i) α_{ijn} = 1 for each $i \in M$, $j \in N$, and ii) α_{ijk_i} = 1 for each $j \in \{N\text{-}N'\}$, $i \in \{M\text{-}W_p\}$ and $1 \le k_i \le n$-1. Given this, for any feasible solution, the r-h-s of *Partial p*-ACM inequality-Case I) will be either n-1 or n. In particular, when y_{ik_i} = 1 for each $i \in W_p$ for some $1 \le k_i \le n\text{-}p$, its r-h-s will be n-1. For such instances, for this inequality to be valid, the maximum value its l-h-s can take cannot exceed n-1. This is accomplished by appropriately selecting a $H_q \subseteq N'$ such that for every feasible solution to (**P**$_{zy}$) in which all agents in W_p are utilized, the assignment of at least one $j \in H_q$ is hidden from every $i \in W_p$, as well as each $i \in \{M\text{-}W_p\}$. It now suffices to specify all the hidden assignments in the *Partial p*-ACM inequality-Case I).

In any feasible solution to (**P**$_{zy}$), define $W_t^- = \{i \in W_p |\ y_{ik_i} = 1$, for some $k_i \in K_i^-\}$ and $W_t^+ = \{i \in W_p |\ y_{ik_i} =$ 1, for some $k_i \in K_i^+\}$. The set of all hidden assignments in the *Partial p*-ACM inequality-Case I) are:

i) The assignment of $j \in \{N\text{-}N'\}$ to $i \in W_p$ with $k_i \in K_i^+$ such that $k_i \le n\text{-}\sum_{i' \in \{W_p - i\}} k(i')$. (54a)

ii) The assignment of a designated $j^- \in H_q$ to each $i \in W_p$, with $k_i \in K_i^-$, and to each $i \in \{M\text{-}W_p\}$ with $1 \le k_i \le n$-1. (54b)

iii) The assignment of a $j \in H_q$ to an $i \in W_p$ with $k_i \in K_i^+$ and to each $i' \in \{W_p\text{-}i\}$ with $k_i \in K_i^-$. (54c)

iv) For each selection of $W_t^- \subset W_p$ with $1 \le |W_t^-| \le p$-2 and $W_t^+ = W_p\text{-}W_t^-$, the assignment of a $j \in H_q$ to each $i \in W_t^-$, $k_i \in K_i^-$, and to each $i^+ \in W_t^+$ whose $k(i^+) \le k_{i^+} \le n\text{-}\sum_{i \in \{W_t^+ - i^+\}} k(i)\text{-}|W_t^-|$. (54d)

v) For each $j \in \{H_q\text{-}j^-\}$ whose assignment to each $i \in W_t^-$ and each $i \in W_t^+$, as specified in (54d), its assignment to each $i \in \{M\text{-}W_p\}$ with $1 \le k_i \le n\text{-}\sum_{i \in W_t^+} k(i)\text{-}|W_t^-|$. (54e)

Since $n' \le \sum_{i \in W_p} k(i) - r$ for some $1 \le r \le p$-1, $n' < \sum_{i \in W_p} k(i)$. Therefore, feasible solutions in (**P**$_{zy}$) in which y_{ik_i} = 1 with $k_i \in K_i^+$ for each $i \in W_p$, at least one $j \in \{N\text{-}N'\}$ must be assigned to an $i \in W_p$. All such assignments are hidden as per (54a). Now consider feasible solutions in (**P**$_{zy}$) in which y_{ik_i} = 1 with $k_i \in K_i^-$ for each $i \in W_p$. As per (54b), the assignment of $j^- \in H_q$ to each $i \in W_p$ with $k_i \in K_i^-$ and each $i \in \{M\text{-}W_p\}$ with $1 \le k_i \le n$-1 is hidden. This ensures that all solutions in which $|W_t^-| = p$, the assignment of $j^- \in H_q$ is hidden. Consider feasible solutions in (**P**$_{zy}$) in which an agent $i \in W_p$ with $k_i \ge n\text{-}p$+1, as well as agents $i' \in \{W_p\text{-}i\}$ with $k_{i'} \in K_{i'}^-$ are utilized. Here, since $\beta_{ik_i} = n - p$ for $k_i \ge n\text{-}p$+1, the r-h-s of the *Partial p*-ACM inequality is n-1. However, (54c) ensures that the l-h-s of the *Partial p*-ACM inequality-Case I) is n-1 as well. Finally, (54d) and (54e) refers to all feasible solutions in (**P**$_{zy}$) not accounted for in (54a), (54b) and (54c). The hidden assignments specified in (54d) and (54e) ensure that the l-h-s of at least one $j \in H_q$ is hidden, resulting in the l-h-s of the *Partial p*-ACM inequality-Case I) being at most n-1.

What is fundamentally different about *Partial p*-ACM inequalities is that, $|H_q| \le n' < 2^p$-1. This implies that, unlike with *Complete p*-ACM inequalities, the assignment of the same $j \in H_q$ has to be hidden in more than one set of feasible solutions. For each $j \in H_q$, let $W^-(j) = \{i \in W_p |\ \alpha_{ijk_i} = 0$, for $k_i \in K_i^-\}$ and $W^+(j) =$ $\{i \in W_p |\ \alpha_{ijk_i} = 0$, for those $k_i \in K_i^+$, in which a feasible solution is possible$\}$. In the *Complete p*-ACM inequality, $W^-(j) \cap W^+(j) = \phi$ for each $j \in H_q$. However, with *Partial p*-ACM inequalities, overlapping sets are introduced for one or more $j \in H_q$, wherein $S(j) = W^-(j) \cap W^+(j) \neq \phi$. For overlapping sets, for each subset $S^-(j) \subseteq S(j)$, the assignment of j to all active agents is hidden in feasible solutions in which: i) y_{ik_i} = 1, for each $i \in \{\{W^-(j)\text{-} S(j)\} \cup S^-(j)\}$ with $k_i \in K_i^-$, and ii) y_{ik_i} = 1, for each $i \in \{\{W^+(j)\text{-}S(j)\} \cup \{S(j) - S^-(j)\}\}$, with $k_i \in K_i^+$.

As with *Complete p*-ACM inequalities, the following set of 'threshold' cardinalities represent the largest cardinality that an agent can be associated with for the assignment of a $j \in H_q$ to be hidden. To satisfy the requirements in (54a), for each $i \in W_p$, $k_{max3}(i) = Max\ \{n\text{-}\sum_{i' \in \{W_p - i\}} k(i'), k(i)\text{-}1\}$. The threshold $k_{max3}(i)$ is the maximum cardinality that can be afforded to i given that each $i' \in \{W_p\text{-}i\}$ has $k(i')$ jobs assigned to it. Provided $k_{max3}(i) \geq k(i)$, the assignment of each $j \in \{N\text{-}N'\}$ to $i \in W_p$ is hidden for $k(i) \leq k_i \leq k_{max3}(i)$, while α_{ijk_i} = 1 for $k_i > k_{max3}(i)$. To satisfy condition (54e), for each $i \in \{M\text{-}W_p\}$ and $j \in \{H_q\text{-}j^-\}$, the threshold $k_{max4}(i,j) = n\text{-}\sum_{i \in \{W_p - W^-(j)\}} k(i)\text{-}|W^-(j)|$, represents the largest feasible cardinality possible for $i \in \{M\text{-}W_p\}$ when each $i \in W^-(j)$ has a cardinality of 1, and each $i \in \{W_p\text{-}W^-(j)\}$ has a cardinality of $k(i)$. Hence, the assignment of each $j \in \{H_q\text{-}j^-\}$ to $i \in \{M\text{-}W_p\}$ with $k_i \leq k_{max4}(i,j)$ is hidden. Next, to satisfy conditions required of each $j \in \{H_q\text{-}j^-\}$ as per (54d), the threshold cardinality, $k_{max5}(i,j)$ is defined. For each $i \in \{W_p\text{-}W^-(j)\}$ and $j \in \{H_q |\ |W^-(j)| \leq p\text{-}2\}$, $k_{max5}(i,j) = Max\ \{n\text{-}\sum_{i' \in \{W_p - W^-(j) - i\}} k(i')\text{-}|W^-(j)|,\ k(i)\text{-}1\}$ represents the maximum cardinality possible with $i \in \{W_p\text{-}W^-(j)\}$, and a $j \in H_q$ whose $|W^-(j)| \leq p$-2. Hence, α_{ijk_i} = 0 for each $j \in \{H_q |\ |W^-(j)| \leq p\text{-}2\}$, $i \in \{W_p\text{-}W^-(j)\}$, $k(i) \leq k_i \leq k_{max5}(i,j)$.

Assuming that sets $W^-(j)$ and $W^+(j)$ for each $j \in H_q$ includes the possibility of $S(j) = \phi$, the generic form of *Partial p*-ACM inequality-Case I) can be stated as:

$$\begin{aligned}
&\sum_{i \in \{M - W_p\}} \sum_{j \in \{N - N'\}} \sum_{k_i = 1}^{n-1} z_{ijk_i} + \sum_{i \in \{M - W_p\}} \sum_{j \in \{H_q - j^-\}} \sum_{k_i = k_{max4}(i,j)+1}^{n-1} z_{ijk_i} + \sum_{i \in W_p} \sum_{k_i = k(i)}^{n-1} z_{ij^-k_i} \\
&\quad + \sum_{i \in \{W_p - W^-(j)\}} \sum_{j \in \{H_q |\ |W^-(j)| \leq p-2\}} \sum_{k_i = k_{max5}(i,j)+1}^{n-1} z_{ijk_i} + \sum_{i \in \{W^-(j) - S(j)\}} \sum_{j \in H_q} \sum_{k_i = k(i)}^{n-1} z_{ijk_i} \\
&\quad + \sum_{i \in \{W^+(j) - S(j)\}} \sum_{j \in H_q} \sum_{k_i \in K_i^-} z_{ijk_i} + \sum_{i \in W_p} \sum_{j \in \{N - N'\}} \sum_{k_i = k_{max3}(i)+1}^{n-1} z_{ijk_i} \\
&\quad + \sum_{i \in W_p} \sum_{j \in \{N' - H_q\}} \sum_{k_i = 1}^{n-1} z_{ijk_i} + \sum_{i \in M} \sum_{j \in N} z_{ijn} \\
&\quad \leq \sum_{i \in W_p} \sum_{k_i = 1}^{n-p} (k_i - 1) y_{ik_i} + \sum_{i \in M} \sum_{k_i = n-p+1}^{n-1} (n-p) y_{ik_i} + \sum_{i \in \{M - W_p\}} \sum_{k_i = 1}^{n-p} k_i y_{ik_i} \\
&\quad + \sum_{i \in M} (n - p + 1) y_{in} + (p - 1). \qquad (55)
\end{aligned}$$

There are numerous possibilities of identifying an appropriate set $H_q \subseteq N'$, and associated sets $W^-(j)$ and $W^+(j)$ for each $j \in H_q$. In the most general case, H_q consists of a mixture, with some associated with overlapping sets, and some not. Algorithmically, an easy heuristic for generating sets, $W^-(j)$, $W^+(j)$ and

$S(j)$, all of the same size, is by 'sliding' them forward in a circular fashion. By this we mean that two adjacent sets differ in their composition by just one entity in W_p. For example, if for a $j_1 \in H_q$, $\{W^-(j_1)$-$S(j_1)\} = \{i_1,.., i_u\}$, $S(j_1) = \{i_{u+1},.., i_{u+s}\}$ and $\{W^+(j_1)$-$S(j_1)\} = \{i_{u+s+1},.., i_p\}$, then an adjacent $j_2 \in H_q$ defines sets $\{W^-(j_2)$-$S(j_2)\} = \{i_2,.., i_{u+1}\}$, $S(j_2) = \{i_{u+2},.., i_{u+s+1}\}$ and $\{W^+(j_2)$-$S(j_2)\} = \{i_{u+s+2},.., i_1\}$. For $l = 3,..,$ p, the sets associated with $j_l \in H_q$ are, $\{W^-(j_l)$-$S(j_l)\} = \{i_l,.., i_{u+l}\}$, $S(j_l) = \{i_{u+l+1},.., i_{u+l+s}\}$ and $\{W^+(j_l)$-$S(j_l)\} = \{i_{u+s+l+1},.., i_{l-1}\}$, while noting the circular nature of the indices. That is, if $x > p$ in i_x, then $x \rightarrow x$-p. Using such an approach, p sets of the same size are generated. Henceforth, we will refer to the above as a 'sliding mechanism'. It needs to be noted that for each $j_l \in H_q$, its assignment is hidden in 2^s feasible solutions where $s = |S(j_l)|$. In the following example, we illustrate the construction of a *Partial p*-ACM inequality-Case I) in which H_q consists of a mixture of jobs, some associated with overlapping sets and some not.

Example 5. Consider the instance, $W_p = \{1,2, 3, 4, 5\}$, M-$W_p = \{6\}$, $N' = \{1,\ldots, 27\}$, $k(1) = 7$, $k(i) = 6$, for $i = 2, \ldots, 5$ and N–$N' = \{28,\ldots, 33\}$. Here, $p = 5$, $r = 4$ and $k_{min} = 6$. Since $n' = 27$, $|H_q| \leq 27$. One construction of H_q consists of $H_q = \{1,\ldots, 26\}$. The first 20 jobs in H_q are associated with non-overlapping sets and the next 5 with overlapping sets. To begin with, $j^- = 26$, for which $|W^-(j)| = 5$ and $|W^+(j)| = 0$. For $j = 1,.., 5$, $|W^-(j)| = 4$, $|W^+(j)| = 1$ and therefore $S(j) = \phi$. For $|W^+(j)| = 1$, there are $5C_1 = 5$ selections of $W^+(j)$ from W_p. For $j = 6,\ldots, 15$, $|W^-(j)| = 3$ and $|W^+(j)| = 2$, comprising of $5C_2 = 10$ selections of $W^+(j)$ from W_p. For $j = 16,\ldots, 20$, $|W^-(j)| = 2$ and $|W^+(j)| = 3$, whose associated sets are obtained using the sliding mechanism described above. They are: i) $W^-(16) = \{1, 2\}$, $W^+(16) = \{3, 4, 5\}$, ii) $W^-(17) = \{2, 3\}$, $W^+(17) = \{4, 5, 1\}$, iii) $W^-(18) = \{3, 4\}$, $W^+(18) = \{5, 1, 2\}$, iv) $W^-(19) = \{4, 5\}$, $W^+(19) = \{1, 2, 3\}$, v) $W^-(20) = \{5, 1\}$, $W^+(20) = \{2, 3, 4\}$. Thus far, the assignment of each $j = 1,\ldots, 20$ is hidden in feasible solutions in which $W_t^- = W^-(j)$, accounting for 20 combinations in which $2 \leq |W^-(j)| \leq 4$. The remaining five combinations for which $|W_t^-| = 2$ comprise of the following overlapping sets. For each $j = 21,\ldots, 25$, the sizes of the sets are $|W^-(j)| = 2$, $|W^+(j)| = 4$, and $|S(j)| = 1$. However, before applying the sliding mechanism, indices in W_p are reordered as $W_p = \{1, 3, 5, 2, 4\}$. That is, the odd numbered elements in the previous order are placed before the even numbered elements. Here, i) $W^-(21) = \{1, 3\}$, $W^+(21) = \{3, 5, 2, 4\}$, ii) $W^-(22) = \{3, 5\}$, $W^+(22) = \{5, 2, 4, 1\}$, iii) $W^-(23) = \{5, 2\}$, $W^+(23) = \{2, 4, 1, 3\}$, iv) $W^-(24) = \{2, 4\}$, $W^+(24) = \{4, 1, 3, 5\}$, v) $W^-(25) = \{4, 1\}$, $W^+(25) = \{1, 3, 5, 2\}$. From these overlapping sets, the remaining five combinations of W_t^- with $|W_t^-| = 2$, as well as all combinations with $|W_t^-| = 1$ are accounted for.

The threshold cardinalities in (55) are as follows. For $j = 1,\ldots, 4$ and $i = 6$, $k_{max4}(6, j)$ = 33-6-4 = 23, and for $j = 5$, $k_{max4}(6, j)$ = 33-7-4 = 22. For all $j \in \{H_q| \, |W^+(j)| = 2$ and $1 \notin W^+(j)\}$, $k_{max4}(6, j)$ = 33-6*2-3 = 18, and for all $j \in \{H_q| \, |W^+(j)| = 2$ and $1 \in W^+(j)\}$, $k_{max4}(6, j)$ = 33-6-7-3 = 17. There are 6 instances in the former and 4 in the latter. For j = 16, 20, 21 and 25, $k_{max4}(6, j)$ = 33-6*3-2 = 13 and for j = 17, 18, 19, 22, 23 and 24, $k_{max4}(6, j)$ = 33-(7+6+6)-2 = 12. For $j = 6,\ldots, 15$ and $i \in W^+(j)$, $k_{max5}(i, j)$ = 33-6-3 = 24, when $i = 1$, else $k_{max5}(i, j)$ = 33-7-3 = 23. For j = 16, 20, 21 and 25, and any $i \in W^+(j)$, $k_{max5}(i, j)$ = 33-(6+6)-2 = 19. For j = 17, 18, 19, 22, 23 and 24, and any $i \in W^+(j)$, $k_{max5}(i, j)$ = 33-(7+6)-2 = 18. For each

$j \in \{N\text{-}N'\}$ and $i = 1$, $k_{max3}(i) = n\text{-}\sum_{i' \in \{W_p - i\}} k(i')$ = 33-4*6 = 9 and for i = 2,…, 5, $k_{max3}(i)$ = 33 – 7 – 3*6 = 8.

We now construct a fractional solution $(z, y)^f \in LP(z, y)$ that violates (55) in the above instance. Let, $M' = W_p$ = {1,…, 5} and N' = {1,…, 27}. As with *Complete p*-ACM inequalities, the fractional part of $(z, y)^f$ is derived by viewing it as flows from nodes in $[M'_{\hat{k}-1}, M'_{\hat{k}}]$ to nodes in N'. Therefore, without loss of generality, integer valued assignments can be made of jobs in $\{N\text{-}N'\}$ to agents in $\{M\text{-}M'\}$. Thus, y_{66} = 1 and z_{6j6} = 1 for j = 28, …, 33. For each $i \in M'$, $\hat{k}_1$ = 7, and $\hat{k}_i$ = 6 for i = 2,…, 5, and accordingly, $y_{i\hat{k}_i-1} = \hat{y}_{K_L} = r/p$ = 4/5 and $y_{i\hat{k}_i} = \hat{y}_{K_U} = (p-r)/p$ = 1/5. The z values associated with each $i \in M'$ and $j \in N'$ are obtained using ***Algorithm_z{M', N'}***, by directing flows from $[M'_{\hat{k}-1}, M'_{\hat{k}}]$ to nodes in N' by completely avoiding hidden assignments. As per step II of ***Algorithm_z{M', N'}***, $\hat{k}_i\text{-}k_{min}$ = 1 for i = 1, and 0 for the rest. Here, $N_s(1)$ = {27} is selected and consequently, $z_{1,27,7} = \hat{y}_{K_U}$ = 1/5, $z_{1,27,6} = \hat{y}_{K_L}$ = 4/5. Post this step, the available capacities, $R\_Cap_{\hat{k}_i-1}$ = 4/5*5 = 4 and $R\_Cap_{\hat{k}_i}$ = 1/5*6 = 6/5.

For each $j \in H_q$, its assignment is hidden from each $i \in W^-(j)$, $k_i = (\hat{k}_i\text{-}1)$ and each $i \in W^+(j)$, $k_i = \hat{k}_i$. Therefore, the flow into these jobs occur only from $i \in W^-(j)$, $k_i = \hat{k}_i$ and $i \in W^+(j)$, $k_i = (\hat{k}_i\text{-}1)$. For j = 1,…, 5, $|W^-(j)|$ = 4, $|W^+(j)|$ = 1 and $S(j) = \phi$. Thus, for j = 1,…, 5, as per step III of ***Algorithm_z{M', N'}***, $z_{ij\hat{k}_i-1} = \hat{y}_{K_L}/(5-4)$ = 4/5, for each $i \in W^+(j)$, $k_i = (\hat{k}_i\text{-}1)$, and $z_{ij\hat{k}_i} = \hat{y}_{K_U}/(5-1)$ = 1/20, for each $i \in W^-(j)$, $k_i = \hat{k}_i$. Consequently, $R\_Cap_{\hat{k}_i-1}$ = 4-4/5 = 16/5 and $R\_Cap_{\hat{k}_i}$ = 6/5 – 4*1/20 = 5/5. Next, for j = 6,…, 15, $|W^-(j)|$ = 3, $|W^+(j)|$ = 2 and $S(j) = \phi$. Therefore, $z_{ij\hat{k}_i-1} = \hat{y}_{K_L}/(5-3)$ = 2/5, for each $i \in W^+(j)$, $k_i = (\hat{k}_i\text{-}1)$, and $z_{ij\hat{k}_i} = \hat{y}_{K_U}/(5-2)$ = 1/15, for each $i \in W^-(j)$, $k_i = \hat{k}_i$. Observe that $4C_1$ = 4 arcs emanate out of each $i \in W^-(j)$ with $k_i = (\hat{k}_i\text{-}1)$, and $4C_2$ = 6 arcs emanate out of each $i \in W^+(j)$, $k_i = \hat{k}_i$, whose end node belongs to {6,…, 15}. Therefore, the remaining capacities get updated as, $R\_Cap_{\hat{k}_i-1} \rightarrow$ 16/5-4*2/5 = 8/5 and $R\_Cap_{\hat{k}_i} \rightarrow$ 5/5 – 6*1/15 = 3/5. For j = 16,…, 20, $|W^-(j)|$ = 2 and $|W^+(j)|$ = 3. Thus, for j = 16,…, 20, the z values obtained are, $z_{ij\hat{k}_i-1} = \hat{y}_{K_L}/3$ = 4/15 for each $i \in W^+(j)$, and $z_{ij\hat{k}_i} = \hat{y}_{K_U}/2$ = 1/10 for each $i \in W^-(j)$. The sets, $W^-(j)$ and $W^+(j)$ for j = 16,…, 20, are obtained using the sliding mechanism. As a result, out of each $i\text{-}(\hat{k}_i\text{-}1) \in M'_{\hat{k}-1}$, three arcs, and out of each $i\text{-}\hat{k}_i \in M'_{\hat{k}}$, two arcs have their end-node at {16,…, 20}. Therefore, the remaining capacities get updated as, $R\_Cap_{\hat{k}_i-1} \rightarrow$ 8/5 – 3*4/15 = 4/5 and $R\_Cap_{\hat{k}_i} \rightarrow$ 3/5 – 2*1/10 = 2/5. Finally, for j = 21,…, 25, since $|W^-(j)|$ = 2, $|W^+(j)|$ = 4, and $|S(j)|$ = 1, its assignment is not hidden from each $i \in \{M'\text{-}W^-(j)\}$, $k_i = \hat{k}_i\text{-}1$, with $|M'\text{-}W^-(j)|$ = 3, and each $i \in \{M'\text{-}W^+(j)\}$, $k_i = \hat{k}_i$, with $|M'\text{-}W^+(j)|$ = 1. Here, $z_{ij\hat{k}_i-1} = \hat{y}_{K_L}/3$ = 4/15 for each $i \in W^+(j)$ and $z_{ij\hat{k}_i} = \hat{y}_{K_U}/1$ = 1/5 for each $i \in W^-(j)$. Here as well, the sets, $W^-(j)$ and $W^+(j)$ are sliding in nature, across j = 21,…, 25. Consequently, $R\_Cap_{\hat{k}_i-1}$ = 4/5 – 3*4/15 = 0 and $R\_Cap_{\hat{k}_i}$ = 2/5 – 1*1/5 = 1/5. The remaining job in H_q is j^* = 27, for which $|W^-(j)|$ = 5 and $|W^+(j)|$ = 0. Therefore, the flow from each $i\text{-}\hat{k}_i \in M'_{\hat{k}}$ to j^* is 1/5. That is, $z_{i27\hat{k}_i}$ = 1/5 for each $i \in M'$, and consequently $R\_Cap_{\hat{k}_i}$= 1/5 – 1/5 = 0. Since $R\_Cap_{\hat{k}_i-1} = R\_Cap_{\hat{k}_i}$ = 0, along with all $j \in N'$ assigned, the resulting fractional solution $(z, y)^f \in LP(z, y)$. More importantly, the fractional flows completely avoid hidden assignments, resulting in the l.h.s of (55) equal t0 33, while its r.h.s equal to 32. This illustrates the maximum possible violation of 1. □

The above example illustrates how a mixture of overlapping and non-overlapping sets associated with each $j \in H_q$ can be used to construct the *Partial p*-ACM inequality-Case I). We now present another example of *Partial p*-ACM inequality-Case I) in which all $j \in H_q$ are associated with overlapping sets. Such *Partial p*-ACM inequalities are useful when *n'* is so small that it cannot accommodate non-overlapping sets. It can be shown that if all jobs in H_q are associated only with overlapping sets, the *Partial p*-ACM inequality-Case I) consists of $|H_q| = lp+1$ jobs for some $1 \leq l \leq \lceil p/2 \rceil$-1. For each s = 1,.., l, there are p jobs in H_q, each associated with overlapping sets $|W^-(j)| = p$-s, $|W^+(j)| = p$-l+s-1 and $|S(j)| = p$-l-1. In addition, each of the the p sets associated with a $j \in H_q$ are generated using the sliding mechanism. The overlapping sets are generated sequentially, starting from s = 1, and ending at s = l. Moving from s-1 to s, for $2 \leq s \leq l$, the indices in W_p are reordered, wherein the odd numbered indices are placed before the even numbered ones. This reordering is executed before generating the sets for stage s. The last job $j^- \in H_q$ is associated with $|W^-(j^-)| = p$ and $|W^+(j^-)| = 0$. The following example illustrates the construction process.

Example 6. Let W_p = {1, 2, 3, 4, 5}, M-W_p = {6}, N' = {1,…, 12}, $k(1)$ = 4, $k(i)$ = 3, for i = 2, …, 5 and N–N' = {13,…, 18}. Since $H_q \subseteq N'$ and $|H_q| = 5l+1$ for some $1 \leq l \leq 3$-1, one possibility is with l = 2, for which $|H_q|$ = 11. Let H_q = {1, …, 10, 12}. Beginning with s = 1, we have that for j = 1,…, 5, $|W^-(j)|$ = 4, $|W^+(j)|$ = 3 and $|S(j)|$ = 2. Given the current ordering in W_p, the contents of the sets associated with j = 1,.., 5, using the sliding mechanism are: i) $W^-(1)$ = {1, 2, 3, 4}, $W^+(1)$ = {3, 4, 5}, ii) $W^-(2)$ = {2, 3, 4, 5}, $W^+(2)$ = {4, 5, 1}, iii) $W^-(3)$ = {3, 4, 5, 1}, $W^+(3)$ = {5, 1, 2}, iv) $W^-(4)$ = {4, 5, 1, 2}, $W^+(4)$ = {1, 2, 3}, v) $W^-(5)$ = {5, 1, 2, 3}, $W^+(5)$ = {2, 3, 4}. Next, for s = 2, the sizes of the overlapping sets are, $|W^-(j)|$ = 3, $|W^+(j)|$ = 4 and $|S(j)|$ = 2. The contents of W_p are reordered as, W_p = {1, 3, 5, 2, 4}. For j = 6, …, 10, using the sliding mechanism, i) $W^-(6)$ = {1, 3, 5}, $W^+(6)$ = {3, 5, 2, 4}, ii) $W^-(7)$ = {3, 5, 2}, $W^+(7)$ = {5, 2, 4, 1}, iii) $W^-(8)$ = {5, 2, 4}, $W^+(8)$ = {2, 4, 1, 3}, iv) $W^-(9)$ = {2, 4, 1}, $W^+(9)$ = {4, 1, 3, 5}, v) $W^-(10)$ = {4, 1, 3}, $W^+(10)$ = {1, 3, 5, 2}. Finally, $W^-(12)$ = {1, 2, 3, 4, 5} and $W^+(12) = \phi$. The threshold cardinalities for this example are as follows. For j = 1, 3, 4 and 5, $k_{max4}(6, j)$ = 18-3-4 = 11, and for j = 2, $k_{max4}(6, j)$ = 18-4-4 = 10. For j = 6, 7 and 8, $k_{max4}(6, j)$ = 18-(3+3)-3 = 9, and for j = 9 and 10, $k_{max4}(6, j)$ = 18-(4+3)-3 = 8. Note that for j = 1,…, 5, $|W^-(j)|$ = 4 = p-1. Therefore, α_{ijk_i} = 0 for i = {W_p-$W^-(j)$}, $k(i) \leq k_i \leq n$-1. For j = 6,…, 10, $k_{max5}(i, j)$ = 18-3-3 = 12, when i = 1 or $1 \in W^-(j)$, else $k_{max5}(i, j)$ = 18-4-3 = 11. For each $j \in \{N\text{-}N'\}$ and i = 1, $k_{max3}(i)$ = n-$\sum_{i' \in \{W_p - i\}} k(i')$ = 18-4*3 = 6 and for i = 2,…, 5, $k_{max3}(i, j)$ = 18 – 4 – 3*3 = 5.

For this example, $(z, y)^f \in LP(z, y)$ that violates (55) can be identified by setting $M' \rightarrow W_p$ and $\hat{k}_i$ = 4 for i = 1, and $\hat{k}_i$ = 3 for i = 2,…, 5. As noted earlier, violation of (55) occurs primarily in the fractional flows from nodes in $[M'_{\hat{k}-1}, M'_{\hat{k}}]$ to nodes in N'. For each $i \in M'$, the fractional y values are, $y_{i\hat{k}_i-1} = \hat{y}_{K_L} = r/p$ = 4/5 and $y_{i\hat{k}_i} = \hat{y}_{K_U} = (p-r)/p$ = 1/5. As per step II of ***Algorithm_z{M', N'}***, $z_{1,11,4} = \hat{y}_{K_U}$ = 1/5, $z_{1,11,3} = \hat{y}_{K_L}$ = 4/5. After this step, the node capacities at each i-$(\hat{k}_i$-1$) \in M'_{\hat{k}-1}$ and i-$\hat{k}_i \in M'_{\hat{k}}$ become, $R\_Cap_{\hat{k}_i-1}$ = 4/5*2 = 8/5 and $R\_Cap_{\hat{k}_i}$ = 1/5*3 = 3/5, respectively. Next, to determine the flow into each j = 1,…, 5, it is noted that their assignment is not hidden from each $i \in \{M'\text{-}W^-(j)\}$ with a cardinality $k_i = \hat{k}_i$-1 and from each $i \in \{M'\text{-}W^+(j)\}$ with a cardinality $k_i = \hat{k}_i$. Since $|W^-(j)|$ = 4 and $|W^+(j)|$ = 3, for j = 1,…, 5, $z_{ij\hat{k}_i-1}$ =

$\hat{y}_{K_L}/(5-4)$ = 4/5 for each $i\in\{M'\text{-}W^-(j)\}$, and $z_{ij\hat{k}_i}$ = $\hat{y}_{K_U}/(5-3)$ = 1/10 for each $i\in\{M'\text{-}W^+(j)\}$. The sets, $W^-(j)$ and $W^+(j)$ for *j* = 1,…, 5, are obtained using a sliding mechanism. Therefore, the number of positive flows from each $i\in M'$, $k_i = \hat{k}_i$-1 and each $i\in M'$, $k_i = \hat{k}_i$, to nodes in {1,…, 5}, is 1 and 2, respectively. Consequently, after this step, $R\_Cap_{\hat{k}_i-1}$ = 8/5 – 4/5 = 4/5 and $R\_Cap_{\hat{k}_i}$ = 3/5 – 2*1/10 = 2/5. The flows into each *j* = 6,…, 10, is based on $|W^-(j)|$ = 3 and $|W^+(j)|$ = 4, for each *j* = 6,…, 10. Therefore, for each *j* = 6,…, 10, $z_{ij\hat{k}_i-1}$ = $\hat{y}_{K_L}/(5-3)$ = 2/5 for each $i\in\{M'\text{-}W^-(j)\}$, and $z_{ij\hat{k}_i}$ = $\hat{y}_{K_U}/(5-4)$ = 1/5 for each $i\in\{M'\text{-}W^+(j)\}$. As s result, $R\_Cap_{\hat{k}_i-1}$ = 4/5 – 2*4/5 = 0 and $R\_Cap_{\hat{k}_i}$ = 2/5 – 1/5 = 1/5 for each $i\in M'$. Finally, for j^- = 12, $W^-(12)$ = {1, 2, 3, 4, 5}. Therefore, flow from each $i\text{-}\hat{k}_i\in M'_{\hat{k}}$ to *j* = 12, gets equally distributed resulting, in $z_{i12\hat{k}_i}$ = 1/5 for each $i\in M'$. Consequently, $R\_Cap_{\hat{k}_i} \rightarrow$ 1/5 – 1/5 = 0, implying that $(z, y)^f\in LP(z, y)$. Since $(z, y)^f\in LP(z, y)$ completely avoids hidden assignments, (55) is violated by the maximum amount of 1. Figure 5 below illustrates the fractional solution part of $(z, y)^f$. □

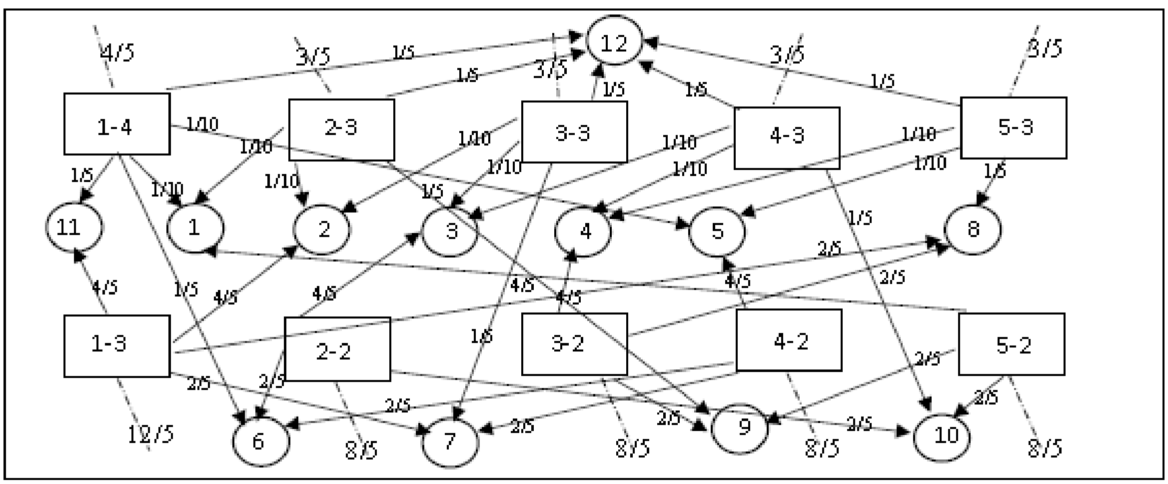

Figure 5. Illustration of the fractional solution in Example 6 in which *p* = 5 and *n'* = 12.

We now address the strength of the *Partial p*-ACM inequality-Case I). Let $\alpha' z \leq \beta' y+(p\text{-}1)$ denote a compact representation of (55). In addition, let *WHK* = {*i*-*j*-k_i| $i\in M$, $j\in N$, 1 ≤ k_i ≤ *n*-1, $\alpha_{ijk_i} = 0$ in (55)} and *H'*(*z*, *y*) = {$(z, y)\in H(z, y)$| $z_{ijk_i} = 0$ for every *i*-*j*-$k_i\in WHK$}. Using a line of argument similar to that presented in the proof of Theorem 3.1 for (43), the following can be shown to hold true for (55).

Theorem 3.2 The *Partial p*-ACM inequality-Case I), (55) is a non-trivial facet of *H'*(*z*, *y*).

Since various *Partial p*-ACM inequalities in the form of (55) can be constructed with varying combinations of overlapping and non-overlapping sets associated with each $j\in H_q$, it need not be a non-trivial facet of *H*(*z*, *y*). This is particularly so if the partitioning cardinalities *k*(*i*) vary across $i\in W_p$. However, it is easy to construct a non-trivial facet of *H*(*z*, *y*) from (55) by sequentially lifting the coefficients of $z_{i'j'k_{i'}}$ whose (i'-j'-$k_{i'}$)$\in WHK$. For each (i'-j'-$k_{i'}$)$\in WHK$, the coefficient $\alpha_{i'j'k_{i'}}$ is determined by solving the optimization problem in (51) in some sequence. Further, it can be shown that the lifted coefficient values will be at most 1. As an illustration, for the *Partial p*-ACM inequality constructed in Example 6, the coefficients of *z* variables after sequentially solving (51) are as follows: α_{362} = α_{562} = α_{272} = α_{562} = α_{282} = α_{482} = α_{193} = α_{492} = $\alpha_{1,10,3}$ = $\alpha_{3,10,2}$ = 1. The coefficients of the remaining *z* variables in *WHK* remain equal to 0.

Note that both in Example 5 and Example 6, $n' = \sum_{i \in W_p} k(i) - r$ with $r = p$-1. Therefore, the fractional solution $(z, y)^f \in LP(z, y)$ that violates (55) was such that $\sum_{i \in W_p} \sum_{k_i=1}^{n-1} y_{ik_i}$=1 for each $i \in W_p$. We now present an example in which $n' \leq \sum_{i \in W_p} k(i) - p$. Therefore, the fractional solution that violates (55) is such that $\sum_{i \in W_p} \sum_{k_i=1}^{n-1} y_{ik_i} < 1$ for each $i \in W_p$.

Example 7. Consider the instance, W_p = {1,2, 3, 4}, M-W_p = {5}, N' = {1,…, 8}, j^- = 7, $k(i)$ = 3, for i = 1, …, 4 and N–N' = {9,…, 13}. Here, p = 4, n' = 8 and β_0 = 3. Since $n' \leq \sum_{i \in W_p} k(i) - p = 8$, a *Partial p*-ACM inequality-Case I) can be constructed with H_q = {1,…, 7}, in which j = 1,.., 4 are each associated with overlapping sets, while j = 5, 6 and 7 are each associated with non-overlapping sets. For j^- = 7, $W^-(7)$ = {1,.., 4} and $W^+(7)$ = ϕ. For j = 1,.., 4, the overlapping sets are: i) $W^-(1)$ = {1, 2, 3}, $W^+(1)$ = {3, 4}, ii) $W^-(2)$ = {2, 3, 4}, $W^+(2)$ = {4, 1}, iii) $W^-(3)$ = {3, 4, 1}, $W^+(3)$ = {1, 2}, iv) $W^-(4)$ = {4, 1, 2}, $W^+(4)$ = {2, 3}. For j = 5, $W^-(5)$ = {1, 3}, $W^+(5)$ = {2, 4}, and for j = 6, $W^-(6)$ = {2, 4}, $W^+(6)$ = {1, 3}. Note that all feasible solutions in which $2 \leq |W_t^-| \leq 3$, the assignment of at least one $j \in$ {1,.., 6} is hidden, while the assignment of j = 7 is hidden in feasible solutions with $|W_t^-|$ = 4. Note as well that feasible solutions in which $|W_t^-| \leq 1$, at least one $j \in \{N$-$N'\}$ will necessarily have to be assigned to a $i \in W_p$ with $k_i \in K_i^+$, whose assignment is hidden.

A fractional solution $(z, y)^f \in H(z, y)$ that violates the *Partial p*-ACM inequality is as follows: i) y_{55} = 1, $z_{5,9,5}$ = $z_{5,10,5}$ = $z_{5,11,5}$ = $z_{5,12,5}$ = $z_{5,13,5}$ = 1, ii) y_{i3} = 1/3 and y_{i2} = 1/2, for i = 1,.., 4, iii) $z_{4,1,2}$ = $z_{1,2,2}$ = $z_{2,3,2}$ = $z_{3,4,2}$ = 1/2, $z_{1,1,3}$ = $z_{2,1,3}$ = $z_{2,2,3}$ = $z_{3,2,3}$ = $z_{3,3,3}$ = $z_{4,3,3}$ = $z_{1,4,3}$ = $z_{4,4,3}$ = 1/4, iv) $z_{2,5,2}$ = $z_{4,5,2}$ = $z_{1,5,3}$ = $z_{3,5,3}$ = 1/4, v) $z_{1,6,2}$ = $z_{3,6,2}$ = $z_{2,6,3}$ = $z_{4,6,3}$ = 1/4, vi) $z_{1,7,3}$ = $z_{2,7,3}$ = $z_{3,7,3}$ = $z_{4,7,3}$ = 1/4, and vii) $z_{1,8,2}$ = $z_{2,8,2}$ = $z_{3,8,2}$ = $z_{4,8,2}$ = 1/4. The l-h-s of the *Partial p*-ACM inequality equals 13 as all the fractional *z* values correspond to assignments which are not hidden. The r-h-s is equal to 12 2/3, resulting in a violation of 1/3. Observe that in this fractional solution, $y_{i2}+y_{i3} < 1$, for each $i \in W_p$. □

Partial p-ACM inequality-Case II)

In *Partial p*-ACM inequality-Case II), the set $N' \subseteq N$ in relation to W_p is even more 'sparse'. Here, $3 \leq |H_q| = n' \leq p$. Since $k_{min} \geq 2$, it follows that if all jobs in N' alone are assigned to agents in W_p, then at most p-1 agents in W_p are active.

Let, H_q be partitioned into two dichotomous sets $H_q(1)$ and $H_q(2)$, with $|H_q(2)| = k_{min}$ and $|H_q(1)| = n'$-k_{min}. Similarly, W_p is partitioned into two dichotomous sets $W_p(1)$ and $W_p(2)$, with $|H_q(1)| = |W_p(1)|$ and $|H_q(2)| \leq |W_p(2)|$. Each $i \in W_p$ is uniquely associated with a $j \in H_q$, denoted as $j(i)$. Thus, for every pair $\{i_1, i_2\} \in W_p(1)$, $j(i_1) \neq j(i_2)$. Given this, the r-h-s parameters of the *Partial p*-ACM inequality-Case II) are, i) $\beta_0 = n'$-k_{min}, ii) $\beta_{in} = n$-β_0 for each $i \in M$, iii) for each $i \in W_p$, β_{ik_i} = 0, for $1 \leq k_i \leq k(i)$-1, β_{ik_i}=$k(i)$-1, for $k(i) \leq k_i \leq n'$, $\beta_{ik_i} = k_i$, for $n'+1 \leq k_i \leq n$-β_0 and β_{ik_i} = n-β_0-1, n-β_0+1 $\leq k_i \leq n$-1, iv) for each $i \in \{M$-$W_p\}$, $\beta_{ik_i} = k_i$ for $1 \leq k_i \leq n$-β_0 and β_{ik_i} = n-n', n-β_0+1 $\leq k_i \leq n$-1. The hidden assignments are as follows. For each $i \in W_p$, $\alpha_{ij(i)k_i}$

= 0 for each $k(i) \le k_i \le n'$. For each $j \in H_q(2)$, i) α_{ijk_i} = 0 for all $i \in W_p$ and $k_i \in K_i^-$, ii) α_{ijk_i} = 0 for all $i \in M$ and $n\text{-}\beta_0+1 \le k_i \le n\text{-}1$, and iii) α_{ijk_i} = 0 for all $i \in \{M\text{-}W_p\}$, $1 \le k_i \le \beta_0\text{-}1$. For each $j \in H_q(1)$, α_{ijk_i} = 0 for all $i \in \{W_p\text{-}j(i)\}$ and $k_i \in K_i^-$. For each $i \in W_p$, $j \in \{N\text{-}N'\}$, α_{ijk_i} = 0 for all $1 \le k_i \le n'$. The *Partial p*-ACM inequality-Case II) obtained is:

$$\sum_{i\in\{M-W_p\}}\sum_{j\in\{N-H_q(2)\}}\sum_{k_i=1}^{n'-k_{min}-1} z_{ijk_i} + \sum_{i\in\{M-W_p\}}\sum_{j\in N}\sum_{k_i=n'-k_{min}}^{n-n'+k_{min}} z_{ijk_i} + \sum_{i\in M}\sum_{j\in\{N-H_q(2)\}}\sum_{k_i=n-n'+k_{min}+1}^{n-1} z_{ijk_i}$$
$$+ \sum_{i\in W_p(1)}\sum_{k_i=1}^{k(i)-1} z_{ij(i)k_i} + \sum_{i\in W_p}\sum_{j\in\{N'-j(i)\}}\sum_{k_i=k(i)}^{n'} z_{ijk_i} + \sum_{i\in W_p}\sum_{j\in N}\sum_{k_i=n'+1}^{n-n'+k_{min}} z_{ijk_i}$$
$$+ \sum_{i\in M}\sum_{j\in N} z_{ijn}$$
$$\le \sum_{i\in W_p}\sum_{k_i=k(i)}^{n'} (k(i)-1)y_{ik_i} + \sum_{i\in\{M-W_p\}}\sum_{k_i=1}^{n'} k_i y_{ik_i} + \sum_{i\in M}\sum_{k_i=n'+1}^{n-n'+k_{min}} k_i y_{ik_i}$$
$$+ \sum_{i\in M}\sum_{k_i=n-n'+k_{min}+1}^{n-1} (n-n')y_{ik_i} + \sum_{i\in M}(n-n'+k_{min})y_{in} + (n'-k_{min}). \quad (56)$$

The above inequality in its current form is not minimal. However, it is made minimal by applying (15a) and (15b) on each $i \in W_p$, $n'+1 \le k_i \le n\text{-}n'+k_{min}$, and on each $i \in \{M\text{-}W_p\}$, $n'\text{-}k_{min} \le k_i \le n\text{-}n'+k_{min}$, to become

$$\sum_{i\in\{M-W_p\}}\sum_{j\in\{N-H_q(2)\}}\sum_{k_i=1}^{n'-k_{min}-1} z_{ijk_i} + \sum_{i\in M}\sum_{j\in\{N-H_q(2)\}}\sum_{k_i=n-n'+k_{min}+1}^{n-1} z_{ijk_i} + \sum_{i\in W_p(1)}\sum_{k_i=1}^{k(i)-1} z_{ij(i)k_i}$$
$$+ \sum_{i\in W_p}\sum_{j\in\{N'-j(i)\}}\sum_{k_i=k(i)}^{n'} z_{ijk_i} + \sum_{i\in M}\sum_{j\in N} z_{ijn}$$
$$\le \sum_{i\in W_p}\sum_{k_i=k(i)}^{n'} (k(i)-1)y_{ik_i} + \sum_{i\in\{M-W_p\}}\sum_{k_i=1}^{n'-k_{min}-1} k_i y_{ik_i}$$
$$+ \sum_{i\in M}\sum_{k_i=n-n'+k_{min}+1}^{n} (n-n')y_{ik_i} + \sum_{i\in M}(n-n'+k_{min})y_{in} + (n'-k_{min}). \quad (57)$$

Proposition 3.3 Every $(z, y) \in H(z, y)$ satisfies (57).

Proof: All feasible integer integer solutions to **(P_{zy})** consist of at least one of the following partial solutions:

I) $(z, y)^{p1}$: y_{ik_i} = 1 for some $i \in M$, $k_i \ge n-n'+k_{min}+1$,

II) $(z, y)^{p2}$: y_{ik_i} = 1 for some $i \in M$, $n'+1 \le k_i \le n-n'+k_{min}$,

III) $(z, y)^{p3}$: y_{ik_i} = 1 for some $i \in W_p$, $k(i) \le k_i \le n'$,

IV) $(z, y)^{p4}$: y_{ik_i} = 1 for some $i \in W_p$, $1 \le k_i \le k(i)\text{-}1$, and

V) $(z, y)^{p5}$: $y_{ik_i} = 1$ for some $i \in \{M\text{-}W_p\}$, $1 \le k_i \le n'\text{-}k_{min}\text{-}1$.

It then suffices to show that feasible solutions that correspond to $(z, y)^{p1}$, $(z, y)^{p2}$, $(z, y)^{p3}$ and $(z, y)^{p4}$, all satisfy (57).

Consider first feasible solutions consisting of $(z, y)^{p1}$. In such solutions, let $V_p \subset \{W_p\text{-}i\}$ and $U \subseteq \{M\text{-}W_p\}$ denote active agents, i.e., $y_{i'k_{i'}}=1$ for each $i' \in V_p$, and $y_{i''k_{i''}}=1$ for each $i'' \in U$. Consider first the situation in which $U = \phi$. If $n'\text{-}k_{min} > k(i')$ for some $i' \in V_p$, then the r-h-s of (57) is at least $(n\text{-}n')+k(i')\text{-}1+(n'\text{-}k_{min}) \ge n\text{-}1$, since $k(i') \ge k_{min}$. Here, since the assignment of $j(i')$ is hidden from i' and i, the l-h-s of (57) is at most n-1 and therefore such solutions satisfy it. If $n'\text{-}k_{min} \le k(i')$ for all $i' \in V_p$, then the r-h-s of (57) is $(n\text{-}n')+(n'\text{-}k_{min}) = n\text{-}k_{min}$. Here, the assignment of all $j \in H_q(2)$ is hidden, and therefore the maximum value the l-h-s of (57) can take is $n\text{-}k_{min}$. Therefore, (57) is satisfied. Next, consider the case where $V_p = \phi$. Here, the r-h-s value of (57) is $(n\text{-}n')+(n'\text{-}k_{min})+\sum_{i'' \in U} k_{i''} = n\text{-}k_{min}+\sum_{i'' \in U} k_{i''}$. The maximum value the l-h-s of (57) can take is $n\text{-}k_{min}$, since only the assignment of all jobs $j \in H_q(2)$ are hidden, (57) is satisfied.

Next, consider feasible solutions that contain $(z, y)^{p2}$. Here, $n'+1 \le k_i \le n\text{-}n'+k_{min}$ for some $i \in M$. The 'contribution' of $(z, y)^{p2}$ to the l-h-s and r-h-s of (57) is zero. Similarly, with $(z, y)^{p5}$, its maximum contribution towards the l-h-s of (57) matches its contribution towards its r-h-s. Hence, validity of (57) does not depend on $(z, y)^{p2}$ or $(z, y)^{p5}$. Finally, consider feasible solutions containing $(z, y)^{p3}$, along with $(z, y)^{p4}$. In such a case, the maximum value the l-h-s of (57) can take, occurs when each $j(i') \in H_q(1)$ is assigned to $i' \in W_p(1)$, while all $j \in H_q(2)$ is assigned to a $i \in W_p(2)$. Here, the assignment of $j(i) \in H_q(2)$ to i is hidden. Therefore, the l-h-s of (57) is n'-1, while the r-h-s is $(k(i)\text{-}1+n'\text{-}k_{min})$. Since $k(i) \ge k_{min}$, (57) is satisfied.□

Using a line of argument similar to that presented in the proof of Theorem 3.1 for (43), the same can be shown to hold true for (57).

Theorem 3.3 The *Partial p*-ACM inequality-Case II) is a non-trivial facet of $H(z, y)$.

The following example illustrates the construction of (57).

Example 8. Let $W_p = \{1, 2, 3, 4\}$, $M\text{-}W_p = \{5\}$, $N' = \{1,\ldots, 4\}$, $k(i) = 2$, for $i = 1, \ldots, 4$, and $N\text{–}N' = \{5,\ldots, 9\}$. Consequently, $k_{min} = |H_q(2)| = 2$, while $\beta_0 = |H_q(1)| = (n' - k_{min}) = 2$. Let $H_q(2) = \{1, 2\}$, $H_q(1) = \{3, 4\}$, $j(1) = 1$, $j(2) = 2$, $j(3) = 3$ and $j(4) = 4$. The inequality (57) that results is:

$$\sum_{j \in \{N-H_q(2)\}} z_{5j1} + \sum_{i \in M} \sum_{j \in \{N-H_q(2)\}} z_{ij8} + z_{341} + z_{441} + \sum_{j \in \{N'-1\}} \sum_{k_i=2}^{4} z_{1jk_i} + \sum_{j \in \{N'-2\}} \sum_{k_i=2}^{4} z_{2jk_i}$$
$$+ \sum_{j \in \{N'-3\}} \sum_{k_i=2}^{4} z_{3jk_i} + \sum_{j \in \{N'-4\}} \sum_{k_i=2}^{4} z_{4jk_i} + \sum_{i \in M} \sum_{j \in N} z_{ij9}$$
$$\le y_{51} + \sum_{i \in W_p} \sum_{k_i=2}^{4} y_{ik_i} + \sum_{i \in M} 5y_{i8} + \sum_{i \in M} 7y_{i9} + 2.$$

A fractional solution $(z, y)^f \in H(z, y)$ that violates the inequality above is: i) $y_{55} = 1$, $z_{555} = z_{565} = z_{575} = z_{585} = z_{595} = 1$, ii) $y_{12} = y_{22} = 1/3$, $y_{32} = y_{42} = 5/12$, $y_{31} = y_{41} = 1/2$, iii) $z_{212} = 1/2$, $z_{312} = z_{412} = 1/4$, $z_{122} = 1/2$, $z_{322} = z_{422} = 1/4$, and iv) $z_{331} = 1/2$, $z_{132} = z_{232} = z_{432} = 1/6$, $z_{441} = 1/2$, $z_{142} = z_{242} = z_{342} = 1/6$. With this fractional solution, the l-h-s of the above inequality is 9, while the r-h-s is 8 1/2, resulting in a violation of 1/2. Figure 6 illustrates the fractional part of $(z, y)^f$. □

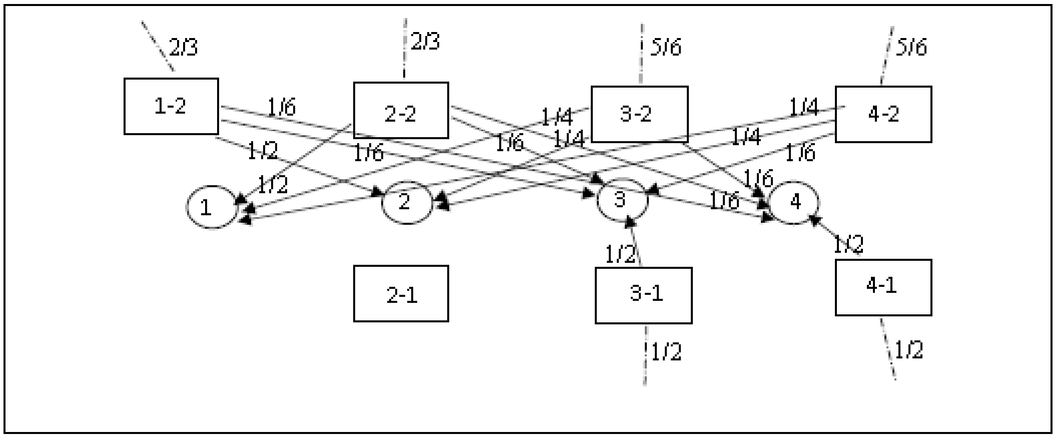

Figure 6. Illustration of the fractional solution in Example 8 in which $p = 4$ and $n' = 4$.

The next example illustrates a fractional solution in which $n' < p$.

Example 9. The problem instance is identical to that in Example 8, except that $M\text{-}W_p = \{4\}$, $N' = \{1,.., 3\}$ and $N\text{-}N' = \{4,\ldots, 8\}$. As before, $k(i) = 2$, for $i = 1, \ldots, 4$, and $k_{min} = |H_q(2)| = 2$. Therefore, $\beta_0 = |H_q(1)| = (n' - k_{min}) = 1$. Let $H_q(2) = \{1, 2\}$, $H_q(1) = \{3\}$, $j(1) = 2$, $j(2) = 2$, $j(3) = 1$ and $j(4) = 3$. The inequality (57) that results is:

$$\sum_{i\in M}\sum_{j\in N} z_{ij8} + z_{341} + \sum_{j\in\{N'-2\}}\sum_{k_i=2}^{3} z_{1jk_i} + \sum_{j\in\{N'-2\}}\sum_{k_i=2}^{3} z_{2jk_i} + \sum_{j\in\{N'-1\}}\sum_{k_i=2}^{3} z_{3jk_i} + \sum_{j\in\{N'-3\}}\sum_{k_i=2}^{3} z_{4jk_i}$$
$$\leq \sum_{i\in W_p}\sum_{k_i=2}^{3} y_{ik_i} + \sum_{i\in M} 7y_{i8} + 1.$$

The following fractional solution that violates the above inequality is: i) $y_{45} = 1$, $z_{445} = z_{455} = z_{465} = z_{475} = z_{485} = 1$, ii) $y_{12} = y_{22} = 1/4$, $y_{32} = y_{42} = 1/2$, and iii) $z_{412} = 1/2$, $z_{112} = z_{212} = 1/4$, $z_{322} = z_{422} = 1/2$, $z_{332} = 1/2$, $z_{132} = z_{232} = 1/4$. This fractional solution violates the above inequality by 1/2, with the l-h-s equal to 8, and the r-h-s equal to 7 ½. □

It is indeed noteworthy that the *Partial p*-ACM inequality-Case II) subsumes the *odd-hole* inequalities of Cornuéjols, G. and Thizy [7] for those cases in which $n' = p$ and odd. To illustrate, consider the instance, $n = n' = p = 3$, with $k(i) = 2$ for $i = 1,\ldots, 3$. One form of the odd-hole inequality that applies in (**P**$_{xy}$) is: $x_{21} + x_{31} + x_{12} + x_{32} + x_{13} + x_{23} \leq y_1 + y_2 + y_3 + 1$. With *Partial p*-ACM inequality-Case II), by setting $\beta_0 = |H_q(1)| = 1$, $H_q(2) = \{1, 2\}$, $H_q(1) = \{3\}$, $j(1) = 1$, $j(2) = 2$ and $j(3) = 3$, the resulting inequality obtained is: $z_{111} + z_{212} + z_{312} + z_{122} + z_{322} + z_{132} + z_{232} \leq y_{12} + y_{22} + y_{32} + 1$. The fractional solution: y_{12} = y_{22} = y_{32} = 1/2, z_{212} =z_{312} = z_{122} = z_{322} = z_{132} = z_{232} = 1/2, violates the above inequality by 1/2, the same as with the odd-hole inequality.

4.0 Concluding Remarks

In this paper, a new extended formulation of the Single-Source Un-capacitated Facility Location Problem (SSUFLP), denoted as ($\mathbf{P}_{zy}$), is presented and studied in depth. This formulation incorporates the notion of cardinality, defined as the number of customers (or jobs) assigned to a facility (or agent). Consequently, the size of this formulation is O(mn^2) as opposed to O(mn) for the traditional formulation. The polytope defined by the convex hull of all feasible solutions to ($\mathbf{P}_{zy}$) is examined. In this study, besides trivial facets, all non-trivial facets are identified, which is shown to be canonical. By this we mean that the coefficients of all variables that describe the assignment of jobs to agents are either 0 or 1. This greatly simplifies the structure of non-trivial facets, which we refer to as *p*-Agent Cardinality Matching (*p*-ACM) inequalities. These inequalities are defined around $N' \subseteq N$ jobs and $W_p \subseteq M$ agents. This in turn is motivated by isolating the fractional part of any feasible solution to the LP relaxation of our extended formulation. That is, all the variables which are non-integer are associated with N' and W_p, which are rendered infeasible by the *p*-ACM inequality.

We present two broad classes of *p*-ACM inequalities: *Complete p*-ACM inequalities and *Partial p*-ACM inequalities. *Complete p*-ACM inequalities apply when $n' \geq 2^p$, while *Partial p*-ACM inequalities apply in cases where $n' \leq 2^p$-1. In addition, two varieties of *Partial p*-ACM inequalities are presented, one in which $p+1 \leq n' \leq 2^p-1$, and the other in which $n' \leq p$. All the inequalities presented are facets of the polytope defined by the convex hull of feasible solutions to ($\mathbf{P}_{zy}$). Clearly then, the *p*-ACM inequalities presented cover all possible combinations of N' and W_p. Therefore, they represent all non-trivial facets of the polytope defined by the convex hull of feasible solutions to ($\mathbf{P}_{zy}$).

In spite of the *p*-ACM inequalities, along with the trivial facets, completely describing the polytope associated with ($\mathbf{P}_{zy}$), two challenges need to be addressed in order to devise an effective *branch-and-cut* strategy. First, as a practical matter, the extended formulation, even if polynomial in size, is too large to be a viable alternative to the traditional formulation, even for reasonable sized problem. One way to overcome this issue is to not incorporate the entire cardinality set of $k_i = 1,\ldots, n$, but a limited cardinality set. This involves solving the LP relaxation of the traditional formulation ($\mathbf{P}_{xy}$) first. Let (x^*, y^*) denote the LP solution obtained. For each $i \in M$, the summation $\sum_{j \in N} x_{ij}^* = n_i^*$ is determined. Typically, n_i^* will be fractional. A relaxed version of ($\mathbf{P}_{zy}$) is constructed in which limited levels of cardinality around $\lfloor n_i^* \rfloor \pm l$. For instance, with $l = 2$, 5 levels of cardinality for each $i \in M$ would be: i) $k_{i1} = \lfloor n_i^* \rfloor$-2, ii) $k_{i2} = \lfloor n_i^* \rfloor$-1, iii) $k_{i3} = \lfloor n_i^* \rfloor$, iv) $k_{i4} = \lfloor n_i^* \rfloor$+1, and v) $k_{i5} = \lfloor n_i^* \rfloor$+2. The relaxed version of ($\mathbf{P}_{zy}$) consists of *z* and *y* variables, each associated with cardinality set: k_{i1}, k_{i2}, k_{i3} and k_{i4}. The resulting formulation is identical to that of (**P**$_{zy}$), except for (7). Here, associated with i_1-k_{i1} and i_4-k_{i4}, the constraints are,

$$\sum_{j \in N} z_{ijk_{i1}} \leq k_i y_{ik_{i1}} \qquad \forall i \in M, \ \ and \tag{58}$$

$$\sum_{j \in N} z_{ijk_{i4}} \geq k_i y_{ik_{i4}} \qquad \forall i \in M. \tag{59}$$

The size of this relaxed version of (**P**$_{zy}$) is O(*lmn*). The other challenge is to devise a separation algorithm to identify an appropriate *p*-ACM inequality that renders a current LP solution infeasible. That is an issue that falls under the realm of future research endevour.

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